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Does a black hole rotate in Chern-Simons modified gravity?

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Rotating black hole solutions in the (3 + 1)-dimensional Chern-Simons modified gravity theory are discussed by taking account of perturbation around the Schwarzschild solution. The zenith-angle dependence of a metric function related to the frame-dragging effect is determined from a constraint equation independently of a choice of the embedding coordinate. We find that at least within the framework of the first-order perturbation method, the black hole cannot rotate for finite black hole mass if the embedding coordinate is taken to be a timelike vector. However, the rotation can be permitted in the limit of $M/r \to 0$ (where M is the black hole mass and r is the radius). For a spacelike vector, the rotation can also be permitted for any value of the black hole mass.

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I. INTRODUCTION

The latest observational results of the cosmic microwave background (CMB) anisotropy from the Wilkinson Microwave Anisotropy Probe (WMAP) [1] are successfully explained by the Λ cold dark matter standard model. However, two big issues still remain: what is dark matter, and what is dark energy? According to the WMAP results, unfortunately about 96% of the contents of the Universe is given by the dark components that we still do not know. Therefore, properties of dark matter and dark energy have eagerly been investigated from observations [2,3].

In contrast with the ordinary approaches [2,3], in which the existence of the dark components is assumed, it is of great interest to investigate alternative gravity theories [4–8] to solve the dark matter and dark energy problem. In this paper, we focus our attention on the Chern-Simons modified gravity theory [9]. This gravity theory was constructed by Deser *et al.* [9] in (2 + 1) spacetime dimensions for the first time by analogy with the topologically massive U(1) and SU(2) gauge theories. The Chern-Simons modified gravity theory was relatively recently extended by Jackiw and Pi [10] to (3 + 1) spacetime dimensions. In the extended theory, the Schwarzschild solution holds without any modification [10]. Therefore, the theory passes the classical tests of general relativity [11]. In this gravity theory, however, the Kerr solution does not hold. Thus, the solution for a rotating black hole should have a different form from the Kerr solution. In (2 + 1)-dimensional Chern-Simons modified gravity, a family of rotating black hole solutions was found by Moussa *et al.* [12]. The solutions have a fascinating feature that observers in this spacetime behave like ones inside the ergosphere of the Kerr spacetime. This feature is similar to that of the rotation of galaxies [13]. Therefore, it is very interesting to investigate rotating black hole solutions in the (3 + 1)-dimensional Chern-Simons modified gravity theory. In this paper, we discuss rotating black hole solutions taking account of perturbation around the Schwarzschild solution.

This paper is organized as follows. In Sec. II, we briefly review the (3 + 1)-dimensional Chern-Simons modified gravity theory. In Sec. III, we consider the perturbation around the Schwarzschild solution to discuss slow rotation of the black hole. First we investigate a constraint equation independently of a choice of the embedding coordinate. In Sec. III A, from the first-order equations of the field equation, we obtain the metric solution taking the embedding coordinate to be timelike. In Sec. III B, we investigate the metric solution for the case in which the embedding coordinate is spacelike. Finally, we provide a summary in Sec. IV. In this paper, we use a unit in which $c = G = 1$.

II. BRIEF REVIEW OF CHERN-SIMONS MODIFIED GRAVITY THEORY

We briefly review the Chern-Simons modification of general relativity developed by Jackiw and Pi [10]. The Chern-Simons modified gravity theory is provided by the action

$$\begin{aligned} I &= \int dx^4 \mathcal{L} = \frac{1}{16\pi} \int dx^4 \Big(\sqrt{-g} R + \frac{1}{4} \vartheta {}^*RR \Big) \\ &= \frac{1}{16\pi} \int dx^4 \Big(\sqrt{-g} R - \frac{1}{2} v_\mu K^\mu \Big), \end{aligned} \tag{1}$$

where the first term in the integrand is the Einstein-Hilbert action, and $v_\mu \equiv \partial_\mu \vartheta$ is an external 4-vector, which is called the embedding coordinate. The Pontryagin density ${}^*RR$ is defined by ${}^*RR \equiv {}^*R^\sigma{}_\tau{}^{\mu\nu} R^\tau{}_{\sigma\mu\nu}$, using the dual Riemann tensor ${}^*R^\tau{}_\sigma{}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} R^\tau{}_{\sigma\alpha\beta}$. The Chern-Simons topological current K^μ is given by

$$K^\mu = \varepsilon^{\mu\alpha\beta\gamma} [\Gamma^\sigma_{\alpha\tau} \partial_\beta \Gamma^\tau_{\gamma\sigma} + \tfrac{2}{3} \Gamma^\sigma_{\alpha\tau} \Gamma^\tau_{\beta\eta} \Gamma^\eta_{\gamma\sigma}], \tag{2}$$

which is related to the Pontryagin density as $\partial_\mu K^\mu = \frac{1}{2} {}^*RR$.

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From the variation of the Lagrange density $\mathcal{L}$ with respect to the metric $g_{\mu\nu}$, it turns out that the field equation has the form

$$G^{\mu\nu} + C^{\mu\nu} = -8\pi T^{\mu\nu}, \tag{3}$$

where $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ is the Einstein tensor, $T^{\mu\nu}$ is the energy-momentum tensor, and $C^{\mu\nu}$ is the Cotton tensor defined as

$$C^{\mu\nu} = -\frac{1}{2\sqrt{-g}}[v_\sigma(\varepsilon^{\sigma\mu\alpha\beta}\nabla_\alpha R^\nu{}_\beta + \varepsilon^{\sigma\nu\alpha\beta}\nabla_\alpha R^\mu{}_\beta) + v_{\tau\sigma}({}^*R^{\tau\mu\sigma\nu} + {}^*R^{\tau\nu\sigma\mu})]. \tag{4}$$

Here $v_{\tau\sigma} \equiv \nabla_\sigma v_\tau = \partial_\sigma\partial_\tau\vartheta - \Gamma^\lambda_{\tau\sigma}\partial_\lambda\vartheta$ is a symmetric tensor. Corresponding to the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ and the equation of motion $\nabla_\mu T^{\mu\nu} = 0$, the following condition should be imposed:

$$0 = \nabla_\mu C^{\mu\nu} = \frac{1}{8\sqrt{-g}}v^{\nu*}RR. \tag{5}$$

This constraint equation implies that diffeomorphism symmetry breaking is suppressed [10].

III. PERTURBATIVE APPROACH TO ROTATING BLACK HOLE SOLUTIONS

In the Chern-Simons modified gravity theory, the Schwarzschild solution as a nonrotating black hole solution holds without any modification as mentioned above. The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6}$$

where M is the black hole mass. This solution gives $C^{\mu\nu} = 0$ and $\nabla_\mu C^{\mu\nu} = v^{\nu*}RR/(8\sqrt{-g}) = 0$ trivially.

In order to take account of rotation of the black hole, let us consider perturbation around the Schwarzschild solution. In the perturbation, the expansion parameter ϵ is related to the angular momentum J of the black hole, i.e., $J \sim O(\epsilon)$. Under the assumption of stationary, axisymmetric spacetime, we can write the form of the perturbed metric as [14–16]

$$ds^2 = -\left(1 - \frac{2M}{r}\right)(1 + h(r,\theta))dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}(1 + m(r,\theta))dr^2 + r^2(1 + k(r,\theta))[d\theta^2 + \sin^2\theta(d\phi - \omega(r,\theta)dt)^2]. \tag{7}$$

The functions $h(r,\theta)$, $m(r,\theta)$, $k(r,\theta)$, and $\omega(r,\theta)$ are of the first order in ϵ. Hereafter, we take account of equations up to the first order in ϵ.

Using this perturbed metric, from the condition (5), we obtain

$$0 = \nabla_\mu C^{\mu\nu} = v^\nu\frac{3M}{r^3}\sin\theta[\omega_{,r\theta} + 2\cot\theta\omega_{,r}], \tag{8}$$

where a subscript comma denotes the partial differentiation with respect to the coordinates. In this expression, the function $\omega(r,\theta)$ only appears. Therefore, we find that solutions for $\omega(r,\theta)$ should have the functional form

$$\omega(r,\theta) = \frac{\varpi(r)}{\sin^2\theta}, \tag{9}$$

where ϖ is a function of r only. While $\omega(r,\theta)$ is singular on the rotation axis ($\theta = 0$ and π), the metric is regular at least up to the first order, because $g_{t\phi} = -r^2\varpi(r) + O(\epsilon^2)$. Note that $g_{t\phi}$ does not vanish on the rotation axis unless $\varpi(r)$ is identically zero. This means that the shift vector $N_i \equiv g_{ti}(i = r,\theta,\phi)$ defined in the (3 + 1) formalism [17] is singular on the rotation axis. It should also be noted that this result is independent of a choice of the embedding coordinate v^ν.

A. Linear perturbation equations and the metric solution for timelike v_μ

We adopt a timelike vector for v_μ, i.e., $v_\mu = (1/\mu_0, 0, 0, 0)$, which is derived from $\vartheta = t/\mu_0$. In our universe, there exists the frame of reference in which the CMB radiation can be seen as an isothermal distribution except for small fluctuations. The frame of reference is specified by a timelike vector. Such a timelike vector is a candidate for the timelike vector v_μ.

From the (tt), (rr), $(r\theta)$, $(r\phi)$, $(\theta\theta)$, $(\theta\phi)$, and $(\phi\phi)$-components of Eq. (3), we can obtain homogeneous differential equations for the functions h, m, and k. Hence, the homogeneous differential equations have a simple solution of $h(r,\theta) = m(r,\theta) = k(r,\theta) = 0$. These differential equations are completely decoupled from the function ω. Since we are now interested in the rotation of the black hole, i.e., the function ω, we do not seek any other solutions for the functions h, m, and k. From the (tr), $(t\theta)$, and $(t\phi)$-components of Eq. (3), we obtain the equations for ω

$$0 = \omega_{,\theta\theta\theta} + r(r - 2M)\omega_{,rr\theta} + 2r(r - 2M)\cot\theta\omega_{,rr} + 5\cot\theta\omega_{,\theta\theta} + 2(2r - 5M)\omega_{,r\theta} + 4(2r - 5M)\cot\theta\omega_{,r} + 3(\cot^2\theta - 1)\omega_{,\theta}, \tag{10}$$

$$0 = r^2(r - 2M)\omega_{,rrr} + r\omega_{,r\theta\theta} + 3r\cot\theta\omega_{,r\theta} + 6r(r - 2M)\omega_{,rr} + 4(r - 3M)\omega_{,r}, \tag{11}$$

$$0 = r(r - 2M)\omega_{,rr} + 4(r - 2M)\omega_{,r} + \omega_{,\theta\theta} + 3\cot\theta\omega_{,\theta}. \tag{12}$$

Here, Eqs. (10) and (11) are obtained, respectively, only from the nonvanishing (tr) and $(t\theta)$-components of the

Cotton tensor, and Eq. (12) is obtained only from the $(t\phi)$-component of the Einstein tensor. Note that these equations do not include μ_0. This is due to a shortcoming of the first-order perturbation method.

From the result of Eq. (9), Eq. (10) is automatically satisfied. From Eqs. (11) and (12), we obtain the differential equations for $\varpi(r)$, respectively,

$$r^2\varpi''' + 6r\varpi'' + 6\varpi' = 0, \tag{13}$$

$$r(r-2M)\varpi'' + 4(r-2M)\varpi' + 2\varpi = 0, \tag{14}$$

where a prime denotes the differentiation with respect to the coordinate r. The solution of Eq. (13) is given by

$$\varpi = C_0 + \frac{C_1}{r} + \frac{C_2}{r^2}, \tag{15}$$

where C_0, C_1, and C_2 are constants of integration. On the other hand, the solution of Eq. (14) is given by

$$\varpi = D_1\frac{r-2M}{r^3} + \frac{D_2}{r^3}[r^2 - 2Mr - 4M^2 + 4M(r-2M) \times \ln(r-2M)], \tag{16}$$

where D_1 and D_2 are constants of integration. Thus, the solution that satisfies both differential equations (13) and (14) is given only by $\varpi(r) = 0$. Therefore, we conclude that within the framework of the first-order perturbation, the black hole cannot rotate in the Chern-Simons modified gravity for the timelike vector.

However, in the limit of $M/r \to 0$, the derivative of Eq. (14) coincides with Eq. (13). Hence, the solution

$$\varpi = \frac{C_1}{r} + \frac{C_2}{r^2}, \tag{17}$$

i.e.,

$$g_{t\phi} = -(C_1 r + C_2), \tag{18}$$

is permitted in this limit. Since the metric component $g_{t\phi}$ is proportional to r at infinity, the frame-dragging effect of this solution works in the whole space.

B. Linear perturbation equations and the metric solution for spacelike v_μ

Next we take another choice of $\vartheta = r\cos\theta/\lambda_0$. This provides a spacelike vector $v_\mu = (0, \cos\theta/\lambda_0, -r\sin\theta/\lambda_0, 0)$, which becomes a unit vector parallel to the rotation axis at infinity. The discrepancy between the observational result and the theoretical prediction in the quadrupole moment of the CMB anisotropy may imply the existence of such a spacelike vector [18].

From the (tt), $(t\phi)$, (rr), $(r\theta)$ $(\theta\theta)$, and $(\phi\phi)$-components of the field equation, we obtain the nonvanishing equations, respectively,

$$\begin{aligned} &2r(r-2M)k_{,rr} - 2(r-2M)m_{,r} + 2(3r-5M)k_{,r} + k_{,\theta\theta} + \cot\theta k_{,\theta} + m_{,\theta\theta} + \cot\theta m_{,\theta} + 2k - 2m \\ &\quad = -\frac{1}{\lambda_0(r-2M)}[r^2(r-2M)^2\varpi''' + r(r-2M)(6r-11M)\varpi'' + 2(r-2M)(3r-2M)\varpi' - 2M\varpi], \end{aligned} \tag{19}$$

$$\begin{aligned} &\frac{1}{\sin^2\theta}[r(r-2M)\varpi'' + 4(r-2M)\varpi' + 2\varpi] \\ &\quad = -\frac{1}{2\lambda_0 r^3}[-r(r-2M)^2(rh_{,rrr} + \cot\theta h_{,rr\theta}) + r(r-2M)^2(rk_{,rrr} + \cot\theta k_{,rr\theta}) - r(r-2M)(h_{,r\theta\theta} - k_{,r\theta\theta}) \\ &\qquad - (r-2M)\cot\theta(h_{,\theta\theta\theta} - k_{,\theta\theta\theta}) - r(r-2M)\{(2r+M)h_{,rr} + (r-3M)m_{,rr} - (3r-2M)k_{,rr}\} \\ &\qquad - (r-2M)\cot\theta\{(3r+M)h_{,r\theta} - (3r-2M)k_{,r\theta} - 3Mm_{,r\theta}\} + (2r-5M)h_{,\theta\theta} - (r-3M)m_{,\theta\theta} - (r-2M)k_{,\theta\theta} \\ &\qquad - (r-2M)\cot^2\theta(h_{,\theta\theta} - k_{,\theta\theta}) + (2r^2 - 9Mr + 10M^2)h_{,r} - 9M(r-2M)m_{,r} \\ &\qquad - 2(r^2 - 9Mr + 14M^2)k_{,r} + 3(r-3M)\cot\theta(h_{,\theta} - m_{,\theta}) + (r-2M)\cot^3\theta(h_{,\theta} - k_{,\theta}) + 2(r-2M)(m-k)], \end{aligned} \tag{20}$$

$$\begin{aligned} &(r-2M)[h_{,\theta\theta} + \cot\theta h_{,\theta} + k_{,\theta\theta} + \cot\theta k_{,\theta} + 2(r-2M)(h_{,r} + k_{,r}) + 2(k-m)] \\ &\quad = \frac{r-3M}{\lambda_0}[r(r-2M)\varpi'' + 4(r-2M)\varpi' + 2\varpi], \end{aligned} \tag{21}$$

$$r(r-2M)(h_{,r\theta} + k_{,r\theta}) - (r-3M)h_{,\theta} - (r-M)m_{,\theta} = -\frac{r\cot\theta}{\lambda_0}[r(r-2M)\varpi'' + 4(r-2M)\varpi' + 2\varpi], \tag{22}$$

$$r(r-2M)(h_{,rr}+k_{,rr})+(r+M)h_{,r}+2(r-M)k_{,r}-(r-M)m_{,r}+\cot\theta(h_{,\theta}+k_{,\theta}) = \frac{\cot^2\theta}{\lambda_0}[r(r-2M)\varpi''+4(r-2M)\varpi'+2\varpi], \tag{23}$$

$$r(r-2M)(h_{,rr}+k_{,rr})+h_{,\theta\theta}+m_{,\theta\theta}+(r+M)h_{,r}+2(r-M)k_{,r}-(r-M)m_{,r} = -\frac{1}{\lambda_0\sin^2\theta}[r(r-2M)\varpi''+4(r-2M)\varpi'+2\varpi-r\{r(r-2M)\varpi'''+2(3r-5M)\varpi''+6\varpi'\}\sin^2\theta]. \tag{24}$$

While the right-hand side of Eq. (22) has the zenith-angle dependence of $\cot\theta$, the left-hand side is composed of the first-order derivatives of h, m, and k with respect to θ. Thus the solution has the form of $(h, m, k) \propto \ln(\sin\theta)$. However, these functions become singular along the rotation axis. Hence, the zenith-angle dependence of the functions h, m, and k should vanish, and therefore these functions depend on r only. On the other hand, this result, i.e., $h = h(r)$, $m = m(r)$, and $k = k(r)$, conflicts with Eq. (23), since the left-hand side becomes a function of r only, and the right-hand side has the dependence of $\cot^2\theta$. Therefore, the functions h, m, and k should vanish. Then, we derive the differential equations

$$0 = r^2(r-2M)^2\varpi'''+r(r-2M)(6r-11M)\varpi'' + 2(r-2M)(3r-2M)\varpi'-2M\varpi, \tag{25}$$

$$0 = r(r-2M)\varpi''+4(r-2M)\varpi'+2\varpi, \tag{26}$$

$$0 = r(r-2M)\varpi'''+2(3r-5M)\varpi''+6\varpi'. \tag{27}$$

Equations (25) and (27) can be derived consistently from Eq. (26). Thus the equation that we have to solve is Eq. (26). In the same way as the case for the timelike v_μ, the differential equation does not include the parameter λ_0. The solution of Eq. (26) is given by the same expression as Eq. (16), which leads to

$$g_{t\phi} = \tilde{D}_1\frac{r-2M}{r}+\frac{\tilde{D}_2}{r}[r^2-2Mr-4M^2 + 4M(r-2M)\ln(r-2M)], \tag{28}$$

where $\tilde{D}_1$ and $\tilde{D}_2$ are constants. Therefore, for the spacelike vector v_μ, the spacetime rotation is permitted for any value of the black hole mass. However, if $\tilde{D}_2 \neq 0$, then the frame-dragging effect extends to infinity, because the second term in Eq. (28) diverges as r increases. Furthermore, the result of Eq. (28) means that the above-mentioned string singularity of the shift vector N_i extends to infinity even if $\tilde{D}_2 = 0$.

IV. SUMMARY

We have investigated the rotation of a black hole in the Chern-Simons modified gravity theory. In particular, we have considered slow rotation of a black hole using the perturbation method, in which the Schwarzschild solution was taken to be the background. From the constraint equation, we obtained the zenith-angle dependence of the metric function $\omega(r, \theta)$ related to the frame-dragging effect, independently of a choice of the embedding coordinate v_μ. Furthermore, by solving the field equation, we found that the black hole cannot rotate for the timelike vector v_μ at least within the framework of the first-order perturbation method. However, in the limit of $M/r \rightarrow 0$, the spacetime rotation is permitted, whose frame-dragging effect extends to infinity. In contrast, for the spacelike vector v_μ, the spacetime rotation is permitted for any value of the black hole mass. Its frame-dragging effect also extends to infinity. Therefore, it is still an open question which form of the metric corresponds to the Kerr solution, which reduces to the Minkowski metric at infinity. Derivation of exact solutions for stationary, axisymmetric spacetimes in the Chern-Simons modified gravity theory may solve this problem. Then, we could also understand effects of the parameter μ_0 or λ_0, which appears in the Chern-Simons term, on the black hole physics. The derivation of exact solutions will be a future work. Furthermore, it should be noted that the above-mentioned results might be modified by the extension of the theory in which ϑ in the Chern-Simons term is taken to be a dynamical variable. This will also be discussed elsewhere.

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