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EFFECTS OF CHERN–SIMONS CORRECTIONS ON CONSERVED QUANTITIES OF RELATIVISTIC FLUID

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We consider relativistic fluid flow under Chern–Simons modified Maxwell theory and under Chern–Simons modified gravity theory. We take account of the effects of Chern–Simons corrections on the quantities of fluid flow that is conserved without the Chern–Simons corrections. We find that the conservations of several quantities are generally broken by the Chern–Simons corrections.

Keywords: Chern–Simons; fluid; Lagrangian description.

1. Introduction

The study of symmetry is requisite in modern physics. A typical example can be found in the standard theory for the interactions between elementary particles. In contrast, the discoveries of symmetry breaking phenomena have opened new frontiers in physics.[1–4] Further detections of such phenomena might provide a signature of new physics. Chern–Simons (CS) modified theories[5,6] are candidates for the theories that describe the breaking of parity symmetry. The CS modified theories have attracted a lot of attentions in the context of exploring new physics.[7–10] The CS modified Maxwell theory[5] is constructed from the usual electromagnetic action with a CS term. Similarly, the CS modified gravity theory[6] is constructed from the Einstein–Hilbert action with a CS term. The CS terms violate the parity symmetry. In previous works, electromagnetic and gravitational fields have mainly been investigated.[11–21] In this paper, we discuss the effect of the symmetry breaking terms on fluid flow under the CS modified theories.

This paper focuses on conserved quantities of fluid. For its discussion, it is useful to adopt the Lagrangian description,[22,23] in which each fluid particle is assigned a label, i.e. Lagrangian coordinates at the initial time. In the Lagrangian coordinates, mass evaluated in a closed region is conserved under time evolution. Similarly, for perfect fluid, circulation is also conserved. These facts are valid even when we take account of general relativity.[22,23] Furthermore, we can encounter other conserved quantities, e.g., fluid helicity,[24–27] magnetic helicity and cross helicity.[28–30,27] The last two quantities are considered for fluid interacting with electromagnetic fields. In this paper, we deal with the above-mentioned conserved quantities within a general relativistic framework and take account of the corrections induced by the CS modified theories.

This paper is organized as follows. In Sec. 2, we consider fluid flow interacting with electromagnetic fields under the CS modified Maxwell theory. Utilizing the Lagrangian description, we discuss the conservation of mass (or energy), circulation, and fluid helicity. We also investigate magnetic helicity and cross helicity. In Sec. 3, we deal with fluid flow under the CS modified gravity theory. We discuss the conserved quantities in the same way as in Sec. 2. Finally we provide a summary in Sec. 4. Throughout the paper, we use geometrized units with $c = G = 1$.

2. Fluid Flow in CS Modified Maxwell Theory

2.1. *CS modified Maxwell theory*

The action of the CS modified Maxwell theory is provided by[5]

$$I_{\mathrm{EM}} = \int d^4 x \left(-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8\pi} v_\mu \,^* F^{\mu\nu} A_\nu \right), \tag{1}$$

where A_μ is the four-potential, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor, $^* F^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}$ is the dual field-strength tensor, and v_μ is an external four-vector called the embedding vector. Here, $\varepsilon^{\mu\nu\lambda\sigma}$ denotes the Levi–Cività tensor with $\varepsilon^{0123} = 1$. Raising and lowering the indices of tensors are done by the Minkowski metric $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$. The second term in the integrand in Eq. (1) is called the CS term. The embedding vector v_μ is now assumed to satisfy $\partial_\mu v_\nu = 0$ to ensure gauge invariance.[5] The dual field-strength tensor satisfies the Bianchi identity $\partial_\mu \,^* F^{\mu\nu} = 0$. The variation of the action with respect to A_μ gives the left-hand side of the electromagnetic field equation

$$\partial_\nu F^{\mu\nu} + v_\nu \,^* F^{\mu\nu} = 4\pi J^\mu \,, \tag{2}$$

where J^μ is the electric four-current. The second term on the left-hand side in this equation stems from the CS term in Eq. (1). Thus the field equation of the Maxwell theory is modified by the CS correction as in Eq. (2).

2.2. *Basic equations for fluid flow*

The basic equations for fluid flow under the CS modified Maxwell theory can be obtained from the conservation law of energy–momentum tensor $T^{\mu\nu}$. The energy–momentum tensor is composed of the fluid part $T_{\mathrm{m}}{}^{\mu\nu}$ and the electromagnetic part $T_{\mathrm{em}}{}^{\mu\nu}$, i.e.

$$T^{\mu\nu} = T_{\mathrm{m}}{}^{\mu\nu} + T_{\mathrm{em}}{}^{\mu\nu}\,. \tag{3}$$

For the fluid part, we assume perfect fluid

$$T_{\mathrm{m}}{}^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + p\eta^{\mu\nu}\,, \tag{4}$$

where ρ is the energy density, p is the pressure, and u^{μ} denotes the four-velocity field of fluid particles ($u^{\mu}u_{\mu} = -1$). In the CS modified Maxwell theory, the electromagnetic part $T_{\mathrm{em}}{}^{\mu\nu}$ is given by[5]

$$T_{\mathrm{em}}{}^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} + \frac{1}{2}v^{\nu}{}^{*}F^{\mu\lambda}A_{\lambda}\right). \tag{5}$$

From the conservation equation $\partial_{\mu}T^{\mu\nu} = 0$, using Eqs. (2), (3) and (5), we obtain

$$\partial_{\mu}T_{\mathrm{m}}{}^{\mu\nu} = F^{\nu\lambda}J_{\lambda}\,. \tag{6}$$

This equation can be divided into two parts, i.e. the component parallel to u^{μ} and the components orthogonal to u^{μ}. The former gives the continuity equation

$$(\rho u^{\mu})_{,\mu} + pu^{\mu}_{\mu} = 0\,, \tag{7}$$

where a comma denotes the partial differentiation with respect to coordinates. The latter gives the equation of motion

$$(\rho + p)u^{\nu}u^{\mu}_{\nu} + P^{\mu\nu}p_{,\nu} = F^{\mu\nu}J_{\nu}\,, \tag{8}$$

where $P^{\mu\nu} \equiv \eta^{\mu\nu} + u^{\mu}u^{\nu}$ is the projection tensor. In deriving Eqs. (7) and (8), we used the ideal magnetohydrodynamics approximation $u_{\mu}F^{\mu\nu} = 0$. We now assume barotropic fluid, for which new variables s and h can be introduced as[31,23]

$$s \equiv \exp\left(\int^{\rho}\frac{d\rho}{\rho + p}\right), \tag{9}$$

$$h \equiv \exp\left(\int^{p}\frac{dp}{\rho + p}\right). \tag{10}$$

The functions s and h correspond to the specific entropy and the specific enthalpy, respectively.[31,23] When the pressure vanishes, we have $s = \rho$ and $h = 1$ by taking appropriate constants of integration in the integrals in Eqs. (9) and (10). By using s and h, Eqs. (7) and (8) are rewritten, respectively, as

$$(su^{\mu})_{,\mu} = 0\,, \tag{11}$$

$$u^\nu u^\mu{}_{,\nu} + P^{\mu\nu}(\ln h)_{,\nu} = N^\mu - V^\mu \,, \tag{12}$$

where

$$N^\mu \equiv \frac{1}{4\pi(\rho + p)} F^\mu{}_\lambda F^{\lambda\sigma}{}_{,\sigma} \,, \tag{13}$$

$$V^\mu \equiv \frac{1}{16\pi(\rho + p)} {}^*F^{\lambda\sigma} F_{\lambda\sigma} v^\mu \,. \tag{14}$$

To derive Eq. (12), we used the identity

$${}^*F^{\mu\lambda} F_{\nu\lambda} = \frac{1}{4}\delta^\mu_\nu F^{\lambda\sigma} F_{\lambda\sigma} \,, \tag{15}$$

where δ^μ_ν denotes the Kronecker delta. In Eq. (12), V^μ comes from the CS correction. Thus while the continuity equation (11) is unchanged, the equation of motion (12) is changed due to the CS correction.

We also discuss vorticity of fluid. For this purpose, let us define the four-vorticity ω^μ as

$$\omega^\mu \equiv \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} u_\nu u_{\lambda,\sigma} \,. \tag{16}$$

The spatial components of the four-vorticity give the usual three-vorticity $\nabla \times \mathbf{v}_{\mathrm{f}}$ when the motion is nonrelativistic, i.e. $u^\mu \simeq (1, v_{\mathrm{f}}{}^i)$ and $|\mathbf{v}_{\mathrm{f}}| \ll 1$, where the Latin index i runs over the spatial coordinates. We emphasize that the temporal component of ω^μ gives the density of fluid helicity $\mathbf{v}_{\mathrm{f}} \cdot (\nabla \times \mathbf{v}_{\mathrm{f}})$ in the nonrelativistic case (see also Refs. 24, 25 and 27). Hence, ω^μ may be regarded as the four-current of fluid helicity. This fact gives us a new insight that we can treat both helicity and circulation of fluid in a unified way by adopting the four-vorticity ω^μ defined in Eq. (16). Furthermore, we can easily recognize the transformation law of helicity density under a coordinate transformation. Under a coordinate transformation $x^\mu \to x'^\mu$, the helicity density ω^0 is transformed according to the transformation law $\omega^\mu \to \omega^{\mu'} = (\partial x'^\mu/\partial x^\nu)\omega^\nu$. From Eq. (12), we obtain the differential equation for ω^μ,

$$\left(\frac{h\omega^\mu}{s}\right)_{,\nu} \frac{u^\nu}{h} - \frac{h\omega^\nu}{s}\left(\frac{u^\mu}{h}\right)_{,\nu}$$

$$= \frac{1}{s}[(N_\nu\omega^\nu u^\mu + {}^*M^{\mu\nu}u_\nu) - (V_\nu\omega^\nu u^\mu + {}^*W^{\mu\nu}u_\nu)] \,, \tag{17}$$

where

$${}^*M^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} N_{\lambda,\sigma} \,, \tag{18}$$

$${}^*W^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} V_{\lambda,\sigma} \,. \tag{19}$$

Here ${}^*M^{\mu\nu}$ and ${}^*W^{\mu\nu}$ are interpreted as rotational parts of the derivatives of N_μ and V_μ, respectively. Thus when an electromagnetic field or the CS correction exists, the vorticity equation becomes an inhomogeneous equation as seen in Eq. (17).

Consequently, the basic equations for fluid flow under CS modified Maxwell theory are given by the field equation (2), the continuity equation (11), the equation of motion (12) and the vorticity equation (17).

2.3. *Lagrangian description of fluid flow*

We deal with the fluid motion from the viewpoint of the Lagrangian description. We adopt the Lagrangian coordinates[22,23]

$$x^\mu = (\tau, x^i) = (\tau, \mathbf{x}),\tag{20}$$

where τ is the proper time of a fluid particle and x^i is constant along a line of fluid flow. In this coordinates, the four-velocity becomes

$$u^\mu = \frac{dx^\mu}{d\tau} = \delta^{0\mu} = (1,0,0,0).\tag{21}$$

Adopting the Lagrangian coordinates is equivalent to taking the four-velocity in the form of Eq. (21). Equation (21) is called the Lagrangian condition.[22,23] When we use the Lagrangian condition, the metric $\eta_{\mu\nu}$ and the partial derivative ∂_μ in Eqs. (2), (11), (12) and (17) must be replaced with the general metric $g_{\mu\nu}$ and the covariant derivative ∇_μ, respectively, because the coordinates are no longer Cartesian. Furthermore, the Levi–Cività tensor is redefined as $\varepsilon^{0123} = 1/\sqrt{-g}$, where g is the determinant of $g_{\mu\nu}$. In the Lagrangian description, the metric is changed with fluid motion, while the coordinates of fluid particles are fixed.

Let us discuss the continuity equation (11) using the Lagrangian condition (21). Equation (11) gives

$$\left(\sqrt{-g}s\right)_{,0} = 0.\tag{22}$$

Thus we obtain

$$s(\tau, \mathbf{x}) = \sqrt{\frac{g(\tau_0, \mathbf{x})}{g(\tau, \mathbf{x})}}s(\tau_0, \mathbf{x}).\tag{23}$$

When the pressure is negligible, Eq. (23) reduces to

$$\rho(\tau, \mathbf{x}) = \sqrt{\frac{g(\tau_0, \mathbf{x})}{g(\tau, \mathbf{x})}}\rho(\tau_0, \mathbf{x}).\tag{24}$$

Therefore, Eqs. (23) and (24) lead to the conservations of entropy density $\sqrt{-g}s$ and energy density $\sqrt{-g}\rho$, respectively, in the Lagrangian coordinates.

Next we discuss the vorticity equation (17) using the Lagrangian condition (21). Here it should be noted that the temporal component of ω^μ is not independent of the spatial components because we have $\omega^0 = g_{0i}\omega^i$ from the equality $u_\mu\omega^\mu = 0$. Hence we focus on the spatial components of ω^μ. The spatial components of Eq. (17) give

$$\left(\frac{h\omega^i}{s}\right)_{,0} = \frac{h}{s}(^*M^{i\nu} - {}^*W^{i\nu})u_\nu.\tag{25}$$

Thus we obtain

$$h(\tau, \mathbf{x}) \omega^i(\tau, \mathbf{x}) = \sqrt{\frac{g(\tau_0, \mathbf{x})}{g(\tau, \mathbf{x})}} \left[h(\tau_0, \mathbf{x}) \omega^i(\tau_0, \mathbf{x}) + s(\tau_0, \mathbf{x}) \right.$$

$$\left. \times \int_{\tau_0}^{\tau} d\tau' \frac{h}{s} (^*M^{i\nu} - {}^*W^{i\nu}) u_\nu \right], \tag{26}$$

where we have used Eq. (23). When the pressure is negligible, Eq. (26) reduces to

$$\omega^i(\tau, \mathbf{x}) = \sqrt{\frac{g(\tau_0, \mathbf{x})}{g(\tau, \mathbf{x})}} \left[\omega^i(\tau_0, \mathbf{x}) + \rho(\tau_0, \mathbf{x}) \int_{\tau_0}^{\tau} d\tau' \frac{1}{\rho} (^*M^{i\nu} - {}^*W^{i\nu}) u_\nu \right]. \tag{27}$$

Equation (26) (or (27)) means that the spatial components $\sqrt{-g}h\omega^i$ (or $\sqrt{-g}\omega^i$) is conserved in the absence of both the rotational part of electromagnetic force and the CS correction. In particular, when the pressure is negligible, we can obtain the conservation of four-vector density $\sqrt{-g}\omega^\mu$ directly from Eq. (17). In such a case, therefore, the conservations of helicity and circulation can be obtained in a unified way within a relativistic framework. In the presence of the rotational part of electromagnetic force or the CS correction, the conservation law concerning vorticity is broken. The integral terms in Eqs. (26) and (27), which are independent of ω^i, become sources of three-vorticity. It means that even if ω^i vanishes everywhere at the initial time, the spatial components of vorticity may be created at some time. The temporal component of the four-vorticity can be derived from $\omega^0 = g_{0i}\omega^i$ as mentioned above. When the pressure vanishes, ω^0 is obtained directly from Eq. (17),

$$\omega^0(\tau, \mathbf{x}) = \sqrt{\frac{g(\tau_0, \mathbf{x})}{g(\tau, \mathbf{x})}} \left[\omega^0(\tau_0, \mathbf{x}) + \rho(\tau_0, \mathbf{x}) \int_{\tau_0}^{\tau} d\tau' \frac{1}{\rho} \right.$$

$$\left. \times \{ (N_\nu \omega^\nu + {}^*M^{0i} u_i) - (V_\nu \omega^\nu + {}^*W^{0i} u_i) \} \right]. \tag{28}$$

The terms $^*M^{0i}u_i$ and $^*W^{0i}u_i$ in the integrand are independent of ω^μ. This fact means that helicity density may also be created irrespective of the values of ω^μ. Therefore, not only the conservation of circulation but also the conservation of helicity is generally broken in the CS modified Maxwell theory.

2.4. Magnetic helicity and cross helicity

Let us discuss magnetic helicity and cross helicity in the CS modified Maxwell theory. These quantities are conserved under certain conditions in ordinary magnetohydrodynamics. In this subsection, we use the Cartesian coordinates, in which the metric reduces to the flat Minkowski metric.

We define the magnetic helicity four-current $H_{\mathrm{m}}{}^\mu$ as

$$H_{\mathrm{m}}{}^\mu \equiv {}^*F^{\mu\nu} A_\nu. \tag{29}$$

Using the identity (15), we derive for ideal magnetohydrodynamics

$$^*F^{\mu\nu}F_{\mu\nu} = -4u_\mu\,^*F^{\mu\lambda}u^\nu F_{\nu\lambda} = 0\,. \tag{30}$$

Thus we obtain

$$H_{\mathrm{m}}{}^\mu{}_{,\mu} = \frac{1}{2}\,^*F^{\mu\nu}F_{\mu\nu} = 0\,. \tag{31}$$

Therefore, the magnetic helicity four-current $H_{\mathrm{m}}{}^\mu$ is conserved even in the CS modified Maxwell theory.

Next we discuss the cross helicity four-current $H_{\mathrm{c}}{}^\mu$ defined by

$$H_{\mathrm{c}}{}^\mu \equiv {}^*\omega^{\mu\nu}A_\nu\,, \tag{32}$$

where $^*\omega^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}u_{\lambda,\sigma}$. For ideal magnetohydrodynamics, we obtain

$$H_{\mathrm{c}}{}^\mu{}_{,\mu} = \frac{1}{2}\,^*\omega^{\mu\nu}F_{\mu\nu} = \frac{1}{2}B^\mu[(\ln h)_{,\mu} + V_\mu]\,, \tag{33}$$

where $B^\mu \equiv {}^*F^{\mu\nu}u_\nu$ denotes the magnetic field. The last term on the right-hand side, which is proportional to $B^\mu V_\mu$, arises from the CS correction. When the right-hand side in Eq. (33) does not vanish, the conservation of cross helicity is broken. Therefore, the cross helicity four-current is not generally conserved in the CS modified Maxwell theory.

3. Fluid Flow in CS Modified Gravity

3.1. *CS modified gravity*

The action of the dynamical CS modified gravity is provided by[6,18]

$$I_{\mathrm{G}} = \int d^4x\sqrt{-g}\left[-\frac{R}{16\pi} + \frac{\ell}{64\pi}\vartheta\,^*R^\tau{}_\sigma{}^{\mu\nu}R^\sigma{}_{\tau\mu\nu} - \frac{1}{2}g^{\mu\nu}(\partial_\mu\vartheta)(\partial_\nu\vartheta) - V(\vartheta) + \mathcal{L}_{\mathrm{m}}\right], \tag{34}$$

where $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ ($R_{\alpha\beta} \equiv R^\lambda{}_{\alpha\lambda\beta}$) is the Ricci scalar, $R^\tau{}_{\sigma\alpha\beta} \equiv \partial_\beta\Gamma^\tau_{\sigma\alpha} - \cdots$ is the Riemann tensor ($\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbols), ℓ is a coupling constant, ϑ is a dynamical scalar field, V is a potential, and $\mathcal{L}_{\mathrm{m}}$ is the Lagrangian density for matter. The dual Riemann tensor is defined by $^*R^\tau{}_\sigma{}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R^\tau{}_{\sigma\alpha\beta}$, where $\varepsilon^{0123} \equiv 1/\sqrt{-g}$. The second term in Eq. (34) is called a Chern–Pontryagin term and connected to the CS term, via partial integration,

$$I_{\mathrm{CS}} = -\frac{\ell}{32\pi}\int d^4x\sqrt{-g}(\partial_\mu\vartheta)\varepsilon^{\mu\alpha\beta\gamma}\left(\Gamma^\sigma_{\alpha\tau}\partial_\beta\Gamma^\tau_{\gamma\sigma} + \frac{2}{3}\Gamma^\sigma_{\alpha\tau}\Gamma^\tau_{\beta\eta}\Gamma^\eta_{\gamma\sigma}\right). \tag{35}$$

Thus nontrivial $\partial_\mu\vartheta$ leads to the CS modification. In this paper, we neglect the problem concerning the surface integral term (see Ref. 32 for the serious treatment).

When we neglect the kinematic term and the potential term of ϑ in Eq. (34), the action reduces to that of the Jackiw–Pi model.[6] From the variations in the action with respect to the metric $g_{\mu\nu}$ and the scalar field ϑ, we obtain the field equations, respectively,

$$G^{\mu\nu} + \ell C^{\mu\nu} = -8\pi(T_{\rm m}{}^{\mu\nu} + T_{\vartheta}{}^{\mu\nu}), \tag{36}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\vartheta = \frac{dV(\vartheta)}{d\vartheta} - \frac{\ell}{64\pi}{}^*R^\tau{}_\sigma{}^{\mu\nu}R^\sigma{}_{\tau\mu\nu}, \tag{37}$$

where $G^{\mu\nu}$ is the Einstein tensor, $C^{\mu\nu}$ is the Cotton tensor defined by

$$C^{\mu\nu} \equiv -\frac{1}{2}[(\nabla_\sigma\vartheta)(\varepsilon^{\sigma\mu\alpha\beta}\nabla_\alpha R^\nu{}_\beta + \varepsilon^{\sigma\nu\alpha\beta}\nabla_\alpha R^\mu{}_\beta) + (\nabla_\sigma\nabla_\tau\vartheta)({}^*R^{\tau\mu\sigma\nu} + {}^*R^{\tau\nu\sigma\mu})], \tag{38}$$

$T_{\rm m}{}^{\mu\nu}$ is the energy–momentum tensor for matter, and $T_{\vartheta}{}^{\mu\nu}$ is the energy–momentum tensor of the scalar field,

$$T_{\vartheta}{}^{\mu\nu} = (\nabla^\mu\vartheta)(\nabla^\nu\vartheta) - g^{\mu\nu}\left[\frac{1}{2}(\nabla^\lambda\vartheta)(\nabla_\lambda\vartheta) + V(\vartheta)\right]. \tag{39}$$

Thus Eqs. (36) and (37) are basic equations in the CS modified gravity.

3.2. *Basic equations for fluid flow*

We obtain basic equations for fluid motion under the CS modified gravity from the covariant divergence of Eq. (36),

$$\nabla_\nu T_{\rm m}{}^{\mu\nu} = -\frac{\ell}{8\pi}\nabla_\nu C^{\mu\nu} - \nabla_\nu T_{\vartheta}{}^{\mu\nu} \equiv \Theta^\mu, \tag{40}$$

where Θ^μ is defined as force exerted on the ordinary matter. We can regard Θ^μ as a CS correction. Using Eq. (39) and the equality[6]

$$\nabla_\nu C^{\mu\nu} = \frac{1}{8}(\nabla^\mu\vartheta){}^*R^\sigma{}_\tau{}^{\nu\lambda}R^\tau{}_{\sigma\nu\lambda}, \tag{41}$$

we obtain

$$\Theta_\mu(\vartheta) = (\nabla^\nu\vartheta)(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\vartheta. \tag{42}$$

Thus we find that Θ_μ vanishes when the scalar field is regular everywhere. Then we derive the usual equation of motion for matter, $\nabla_\nu T_{\rm m}{}^{\mu\nu} = 0$. In this case, the motion of fluid is affected only through the change of the metric. When we consider a test particle, we derive the usual geodesic equation from $\nabla_\nu T_{\rm m}{}^{\mu\nu} = 0$, which is independent of the mass. This fact means that the equivalence principle is valid for regular ϑ under the CS modified gravity. On the other hand, if there is a singularity in ϑ, Θ_μ would not vanish at the singularity. For example, if $\vartheta = \arctan(y/x) \equiv \phi$ in rectangular Cartesian coordinates, we derive

$$\Theta_\mu = \left(0, \frac{2\pi x}{x^2 + y^2}\delta(x)\delta(y), \frac{2\pi y}{x^2 + y^2}\delta(x)\delta(y), 0\right), \tag{43}$$

where we have used the formula[33]

$$(\partial_x\partial_y - \partial_y\partial_x)\phi = 2\pi\delta(x)\delta(y)\,. \tag{44}$$

Here, $\delta(x)$ denotes the Dirac's delta function. Therefore, when such a topological singularity exists, the force term Θ^μ plays an important role at the singularity. Hereafter, we take account of the effect of the CS correction Θ^μ on fluid motion. We now assume perfect fluid again. Equation (40) can be divided into two parts, i.e. the component parallel to u^μ and the components orthogonal to u^μ. The former gives the balance equation, i.e. the continuity equation with a source term,

$$(\rho u^\mu)_{;\mu} + pu^\mu{}_{;\mu} = -u_\mu\Theta^\mu\,, \tag{45}$$

where a semicolon denotes the covariant derivative. The latter gives the equation of motion

$$(\rho + p)u^\nu u^\mu{}_{;\nu} + P^{\mu\nu}p_{,\nu} = P^\mu{}_\nu\Theta^\nu\,, \tag{46}$$

where $P^{\mu\nu}$ is the projection tensor onto the hypersurface orthogonal to u^μ. Using the variables s and h, we can write Eqs. (45) and (46), respectively, as

$$(su^\mu)_{;\mu} = -\mathcal{S}\,, \tag{47}$$

$$u^\nu u^\mu{}_{;\nu} + P^{\mu\nu}(\ln h)_{,\nu} = \mathcal{V}^\mu\,, \tag{48}$$

where the CS corrections $\mathcal{S}$ and $\mathcal{V}^\mu$ are defined by

$$\mathcal{S} \equiv \frac{s}{\rho + p}u_\mu\Theta^\mu\,, \tag{49}$$

$$\mathcal{V}^\mu \equiv \frac{1}{\rho + p}P^\mu{}_\nu\Theta^\nu\,. \tag{50}$$

The balance equation (47) and the equation of motion (48) govern fluid dynamics under the CS modified gravity.

From Eq. (48), we also obtain the differential equation for vorticity ω^μ as

$$\left(\frac{h\omega^\mu}{s}\right)_{;\nu}\frac{u^\nu}{h} - \frac{h\omega^\nu}{s}\left(\frac{u^\mu}{h}\right)_{;\nu} = \frac{1}{s^2}\mathcal{S}\omega^\mu + \frac{1}{s}[\mathcal{V}_\nu\omega^\nu u^\mu + {}^*\mathcal{W}^{\mu\nu}u_\nu]\,, \tag{51}$$

where

$${}^*\mathcal{W}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\mathcal{V}_{\lambda;\sigma}\,. \tag{52}$$

Here ${}^*\mathcal{W}^{\mu\nu}$ is regarded as the rotational component of the force $\mathcal{V}^\mu$. Thus, Eq. (51) governs the time evolution of vorticity in fluid motion under the CS modified gravity.

3.3. *Lagrangian description of fluid flow*

We discuss fluid motion from the viewpoint of the Lagrangian description.

We deal with the balance equation (47). Applying the Lagrangian condition (21) to Eq. (47), we derive

$$\left(\sqrt{-g}s\right)_{,0} = -\sqrt{-g}\mathcal{S} \,. \tag{53}$$

Thus we obtain

$$s(\tau, \mathbf{x}) = \sqrt{\frac{g(\tau_0, \mathbf{x})}{g(\tau, \mathbf{x})}}s(\tau_0, \mathbf{x}) - \frac{1}{\sqrt{-g(\tau, \mathbf{x})}}\int_{\tau_0}^{\tau}d\tau'\sqrt{-g}\mathcal{S} \,. \tag{54}$$

When the pressure vanishes, Eq. (54) reduces to

$$\rho(\tau, \mathbf{x}) = \sqrt{\frac{g(\tau_0, \mathbf{x})}{g(\tau, \mathbf{x})}}\rho(\tau_0, \mathbf{x}) - \frac{1}{\sqrt{-g(\tau, \mathbf{x})}}\int_{\tau_0}^{\tau}d\tau'\sqrt{-g}\mathcal{S} \,. \tag{55}$$

From Eqs. (54) and (55), we see that if the CS correction $\mathcal{S}$ vanishes, the entropy density $\sqrt{-g}s$ or the energy density $\sqrt{-g}\rho$ is conserved in the Lagrangian coordinates. However, if the CS correction $\mathcal{S}$ does not vanish, $\sqrt{-g}s$ and $\sqrt{-g}\rho$ are no longer conserved. The function $\mathcal{S}$ provides a source of entropy or energy as seen in Eqs. (54) and (55). Thus in general, the creation of entropy or energy may occur when there is a topological singularity in the scalar field appearing in the CS modified gravity.

Next we discuss the vorticity equation (51) using the Lagrangian condition (21). As mentioned above, the temporal component of ω^{μ} is not independent of the spatial components because $\omega^{0} = g_{0i}\omega^{i}$. Hence we focus on the spatial components of ω^{μ} again. The spatial components of Eq. (51) give

$$\left(\frac{h\omega^{i}}{s}\right)_{,0} = \frac{h}{s^{2}}\mathcal{S}\omega^{i} + \frac{h}{s}\,{}^{*}\mathcal{W}^{i\nu}u_{\nu} \,. \tag{56}$$

Then we obtain

$$h(\tau, \mathbf{x})\omega^{i}(\tau, \mathbf{x}) = \frac{s(\tau, \mathbf{x})}{s(\tau_0, \mathbf{x})}h(\tau_0, \mathbf{x})\omega^{i}(\tau_0, \mathbf{x}) + s(\tau, \mathbf{x})\int_{\tau_0}^{\tau}d\tau'\left(\frac{h}{s^{2}}\mathcal{S}\omega^{i} + \frac{h}{s}\,{}^{*}\mathcal{W}^{i\nu}u_{\nu}\right). \tag{57}$$

When the pressure vanishes, Eq. (57) reduces to

$$\omega^{i}(\tau, \mathbf{x}) = \frac{\rho(\tau, \mathbf{x})}{\rho(\tau_0, \mathbf{x})}\omega^{i}(\tau_0, \mathbf{x}) + \rho(\tau, \mathbf{x})\int_{\tau_0}^{\tau}d\tau'\left(\frac{1}{\rho^{2}}\mathcal{S}\omega^{i} + \frac{1}{\rho}\,{}^{*}\mathcal{W}^{i\nu}u_{\nu}\right). \tag{58}$$

From Eqs. (57) and (58), we see that if the CS corrections $\mathcal{S}$ and ${}^{*}\mathcal{W}^{i0}$ vanish, $h\omega^{i}/s$ or ω^{i}/ρ is conserved in the Lagrangian coordinates. However, if $\mathcal{S} \neq 0$ or ${}^{*}\mathcal{W}^{i\nu} \neq 0$, $h\omega^{i}/s$ or ω^{i}/ρ is no longer conserved. In particular, the second terms of the integrands in Eqs. (57) and (58) may produce vorticity, regardless of the

values of vorticity vector. Therefore, even if the vorticity vanishes everywhere at the initial time, circulation may be created when there is a topological singularity in the scalar field of the CS modified gravity.

4. Summary

We have considered fluid flow under the CS modified Maxwell theory and under the CS modified gravity theory. We investigated the effects of the CS corrections on conserved quantities of relativistic fluid. First of all, we obtained the CS corrections to the equations for fluid, i.e. the continuity equation, the equation of motion and the vorticity equation. For the discussion of vorticity, we introduced the four-vorticity in Eq. (16). We also pointed out that the four-vorticity provides the expressions of both three-vorticity and fluid helicity density in the nonrelativistic limit. This fact means that the four-vorticity unifies the physical pictures of circulation and fluid helicity in a relativistic framework. To discuss conserved quantities of fluid, we utilized the Lagrangian description of fluid flow. In the CS modified Maxwell theory, we found that while the entropy or energy is conserved, the circulation and fluid helicity are not generally conserved. We also found that the conservation of the magnetic helicity holds even in the CS modified Maxwell theory, while the conservation of the cross helicity does not hold in general. In the CS modified gravity, when the CS scalar field is regular everywhere, the CS corrections do not appear in the equations for fluid flow. In this case, we have the validity of the equivalence principle. This fact means that for a regular CS scalar field, we cannot find the CS correction directly from the test of equivalence principle. Furthermore, we found that when there is a topological singularity in the CS scalar field, not only the circulation and fluid helicity but also the entropy and energy are not conserved in general at the singularity.

Acknowledgments

References

1. S. L. Adler, *Phys. Rev.* **177**, 2426 (1969).
2. J. S. Bell and R. Jackiw, *Nuovo Cimento A* **60**, 47 (1969).
3. P. W. Anderson, *Phys. Rev.* **130**, 439 (1963).
4. See also S. Weinberg, *The Quantum Theory of Fields* (Cambridge Univ. Press, 1996), §21.6.
5. S. M. Carroll, G. B. Field and R. Jackiw, *Phys. Rev. D* **41**, 1231 (1990).
6. R. Jackiw and S.-Y. Pi, *Phys. Rev. D* **68**, 104012 (2003).
7. A. Lue, L. Wang and M. Kamionkowski, *Phys. Rev. Lett.* **83**, 1506 (1999).
8. B. Feng, M. Li, J.-Q. Xia, X. Chen and X. Zhang, *Phys. Rev. Lett.* **96**, 221302 (2006).

9. J.-Q. Xia, H. Li, G.-B. Zhao and X. Zhang, *Astrophys. J.* **679**, L61 (2008).
10. J.-Q. Xia, H. Li and X. Zhang, arXiv:0908.1876.
11. S. M. Carroll and G. B. Field, *Phys. Rev. D* **43**, 3789 (1991).
12. S. M. Carroll and G. B. Field, *Phys. Rev. Lett.* **79**, 2394 (1997).
13. S. M. Carroll, *Phys. Rev. Lett.* **81**, 3067 (1998).
14. G. B. Field and S. M. Carroll, *Phys. Rev. D* **62**, 103008 (2000).
15. S. Alexander and N. Yunes, *Phys. Rev. D* **75**, 124022 (2007).
16. S. Alexander and N. Yunes, *Phys. Rev. Lett.* **99**, 241101 (2007).
17. K. Konno, T. Matsuyama and S. Tanda, *Phys. Rev. D* **76**, 024009 (2007).
18. T. L. Smith, A. L. Erickcek, R. R. Caldwell and M. Kamionkowski, *Phys. Rev. D* **77**, 024015 (2008).
19. K. Konno, T. Matsuyama, Y. Asano and S. Tanda, *Phys. Rev. D* **78**, 024037 (2008).
20. K. Konno, T. Matsuyama and S. Tanda, *Prog. Theor. Phys.* **122**, 561 (2009).
21. See for a comprehensive review of the CS modified gravity, S. Alexander and N. Yunes, *Phys. Rep.* **480**, 1 (2009).
22. H. Asada and M. Kasai, *Phys. Rev. D* **59**, 123515 (1999).
23. H. Asada, *Phys. Rev. D* **62**, 127301 (2000).
24. J. J. Mareau, *C. R. Acad. Sci. Paris* **252**, 2810 (1961).
25. H. K. Moffatt, *J. Fluid Mech.* **35**, 117 (1969).
26. See for a review, Y. Fukumoto, *Topologica* **1**, 003 (2008).
27. See for a relativistic extension, J. D. Bekenstein, *Astrophys. J.* **319**, 207 (1987).
28. L. Woltjer, *Ann. NY Acad. Sci.* **257**, 76 (1956).
29. L. Woltjer, *Proc. Nat. Acad. Sci.* **44**, 489 (1958).
30. L. Woltjer, *Proc. Nat. Acad. Sci.* **44**, 833 (1958).
31. J. Ehlers, *Gen. Rel. Grav.* **25**, 1225 (1993).
32. D. Grumiller, R. Mann and R. McNees, *Phys. Rev. D* **78**, 081502 (2008).
33. R. Jackiw and S.-Y. Pi, *Phys. Rev. Lett.* **66**, 2682 (1991).