Computational Fluids Mechanics

Part 1：Numerical methods
Part 2：Turbulence models
Part 3：Practice of numerical simulation

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Computational Fluid Mechanics
Part1：Numerical Methods

Objective：

Numerical methods for fluids mechanics and their accuracy

1. Introduction
2. Numerical methods for fluid mechanics (1) ~ Basic equations
3. Numerical methods for fluid mechanics (2) ~Discretizing schemes
4. Numerical methods for fluid mechanics (3) ~Coupling algorism
5. Numerical methods for fluid mechanics (4) ~Additional problems
6. Reliability of numerical simulation

Summary

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Computational Fluid Mechanics

Part1： Numerical Methods (3a)

1. Introduction
2. Numerical methods for fluid mechanics (1) ~ governing equations
3. Numerical methods for fluid mechanics (2) ~Discretizing schemes
   - Fundamentals for discretizing methods
   - Schemes for convection and diffusion equation
   - Analysis of accuracy and instability
4. Numerical methods for fluid mechanics (3) ~Coupling algorism
5. Numerical methods for fluid mechanics (4) ~Additional problems
6. Reliability of numerical simulation

Fundamentals for discretizing methods

Generalized conservation eq.

$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \nabla \phi - \nabla \cdot \mathbf{j} + S \qquad \mathbf{j} = -\Gamma \nabla \phi$$

$$\frac{\partial \phi}{\partial t} = -u_j \frac{\partial \phi}{\partial x_j} - \frac{\partial J_j}{\partial x_j} + S \qquad J_j = -\Gamma \frac{\partial \phi}{\partial x_j}$$

$$\frac{\partial \phi}{\partial t} = -\left(u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial x} + w \frac{\partial \phi}{\partial z} \right) - \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right) + S$$

$$\mathbf{j}^t = \begin{bmatrix} J_x & J_y & J_z \end{bmatrix} = \begin{bmatrix} -\Gamma \frac{\partial \phi}{\partial x} & -\Gamma \frac{\partial \phi}{\partial z} & -\Gamma \frac{\partial \phi}{\partial z} \end{bmatrix}$$

Fundamentals for discretizing methods

Finite Different Method

Differential eq. → Polynomial eq.

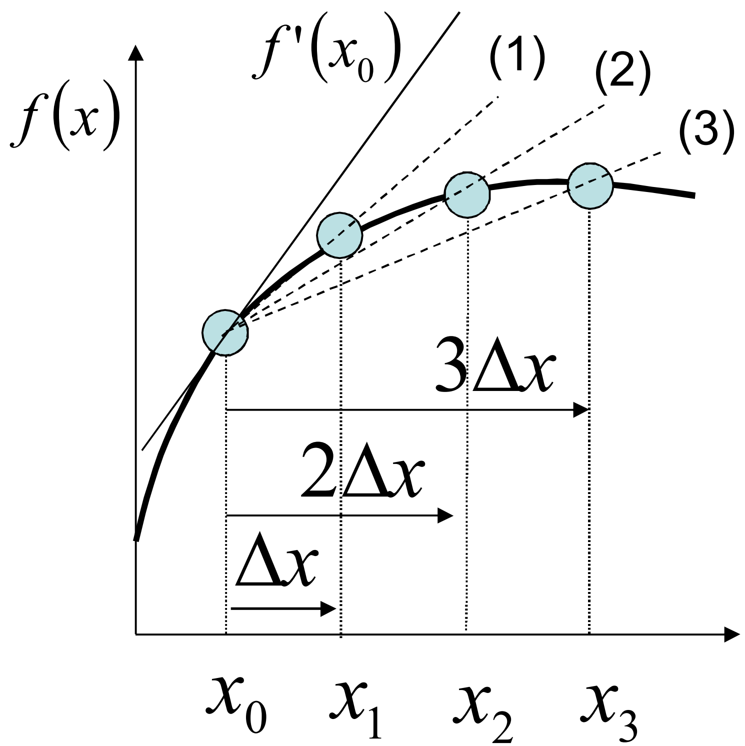
$$\frac{df}{dx} = g(x) \quad \Rightarrow \quad \frac{f(x_1) - f(x_0)}{x_1 - x_0} = g(x_0)$$

$$f'(x_0) \equiv \lim_{x=x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$\cong \frac{f(x_1) - f(x_0)}{\Delta x} \qquad (1)$$

$$\cong \frac{f(x_2) - f(x_0)}{2\Delta x} \qquad (2)$$

$$\cong \frac{f(x_3) - f(x_0)}{3\Delta x} \qquad (3)$$



Taylor series expansion

Taylor series expansion

$$f = \sum \frac{1}{k!} x^k \frac{\partial^k f}{\partial x^k} = f_i + x f'_i + \frac{1}{2} x^2 f''_i + \frac{1}{6} x^3 f'''_i + \frac{1}{24} x^4 f''''_i + \cdots$$

⇩

$$f_{i\pm1} = f_i \pm \delta x f'_i + \frac{1}{2} \delta x^2 f''_i \pm \frac{1}{6} \delta x^3 f'''_i + \frac{1}{24} \delta x^4 f''''_i + \cdots$$

⇩

1st order forward scheme

truncation error

$$\frac{f_{i+1} - f_i}{\delta x} \cong f'_i \qquad + \frac{1}{2} \delta x f''_i + \frac{1}{6} \delta x^2 f'''_i + \frac{1}{24} \delta x^3 f''''_i + \cdots$$

1st order backward scheme

$$\frac{f_i - f_{i-1}}{\delta x} \cong f'_i \qquad - \frac{1}{2} \delta x f''_i + \frac{1}{6} \delta x^2 f'''_i - \frac{1}{24} \delta x^3 f''''_i + \cdots$$

2nd order central scheme

$$f_{i+1} = f_i + \delta x f'_i + \frac{1}{2}\delta x^2 f''_i + \frac{1}{6}\delta x^3 f'''_i + \frac{1}{24}\delta x^4 f''''_i + \cdots$$

$$-\big]\quad f_{i-1} = f_i - \delta x f'_i + \frac{1}{2}\delta x^2 f''_i - \frac{1}{6}\delta x^3 f'''_i + \frac{1}{24}\delta x^4 f''''_i + \cdots$$

$$f_{i+1} - f_{i-1} = 2\delta x f'_i + \frac{1}{3}\delta x^3 f'''_i + \cdots$$

$$\Rightarrow \quad \frac{f_{i+1} - f_{i-1}}{2\delta x} \cong f'_i \qquad + \frac{1}{6}\delta x^2 f'''_i + \cdots$$

1[st] derivative approx. — truncation error

$$f_{i+1} = f_i + \delta x f'_i + \frac{1}{2}\delta x^2 f''_i + \frac{1}{6}\delta x^3 f'''_i + \frac{1}{24}\delta x^4 f''''_i + \cdots$$

$$+\big]\quad f_{i-1} = f_i - \delta x f'_i + \frac{1}{2}\delta x^2 f''_i - \frac{1}{6}\delta x^3 f'''_i + \frac{1}{24}\delta x^4 f''''_i + \cdots$$

$$f_{i+1} + f_{i-1} = 2 f_i + \delta x^2 f''_i + \frac{1}{12}\delta x^4 f''''_i + \cdots$$

$$\Rightarrow \quad \frac{f_{i+1} - 2 f_i + f_{i-1}}{\delta x^2} \cong f''_i \qquad + \frac{1}{12}\delta x^2 f''''_i + \cdots$$

2nd derivative approx. — truncation error

Exercise

$$f'_i \cong \frac{3f_i - 4f_{i-1} + f_{i-2}}{2\delta x}$$

Polynomial factors:

$$\frac{3}{2\delta x}\times \qquad -\frac{4}{2\delta x}\times \qquad +\frac{1}{2\delta x}\times$$

Taylor series expansion:

$$f_i = f_i$$

$$f_{i-1} = f_i - \delta x\, f'_i + \frac{1}{2}\delta x^2\, f''_i - \frac{1}{6}\delta x^3\, f'''_i + \cdots$$

$$f_{i-2} = f_i - (2\delta x) f'_i + \frac{1}{2}(2\delta x)^2 f''_i - \frac{1}{6}(2\delta x)^3 f'''_i + \cdots$$

Look here!

$$\frac{3f_i - 4f_{i-1} + f_{i-2}}{2\delta x} = \left(\frac{3}{2\delta x} - \frac{4}{2\delta x} + \frac{1}{2\delta x}\right) f_i = 0 \times f_i$$

$$+\left(+\frac{4}{2\delta x}\delta x - \frac{1}{2\delta x}(2\delta x)\right) f'_i = 1 \times f'_i$$

$$+\left(-\frac{4}{2\delta x}\frac{(\delta x)^2}{2} + \frac{1}{2\delta x}\frac{(2\delta x)^2}{2}\right) f''_i = 0\,\delta x \times f''_i$$

$$+\left(+\frac{4}{2\delta x}\frac{(\delta x)^3}{6} - \frac{1}{2\delta x}\frac{(2\delta x)^3}{6}\right) f'''_i = -\frac{1}{3}\delta x^2 \times f'''_i$$

$1 \times f'_i$ → Approx.

Truncation error

$-\frac{1}{3}\delta x^2 \times f'''_i$ → 2nd order accuracy

Fundamentals for discretizing methods

Numerical integration

(for Finite Element Method / Time marching scheme)

Differential eq.

$$\frac{df}{dx} = g(x, f(x))$$

⇨

Integral eq.

$$[f]_{x_0}^{x} \equiv f(x) - f(x_0) = \int_{x_0}^{x} g(x')dx'$$

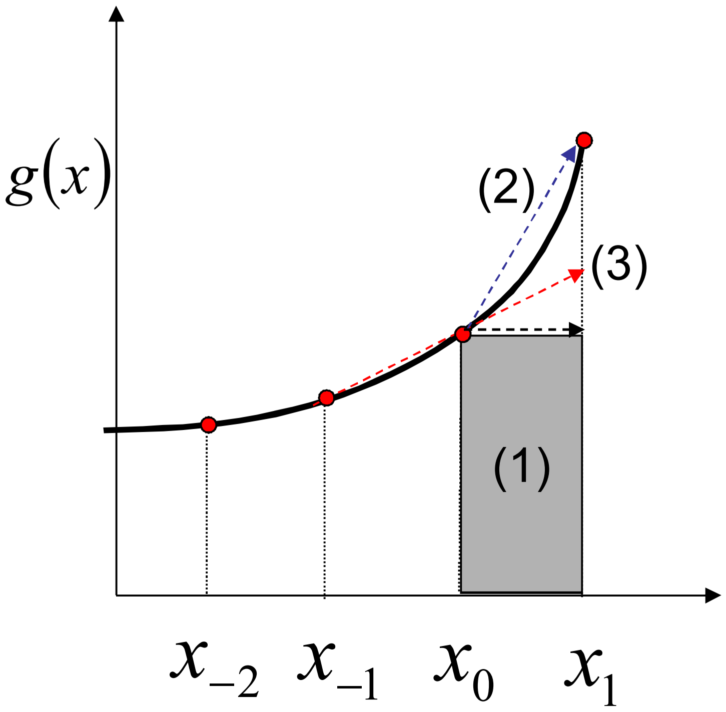
Numerical integration of (g)

$$f(x_1) - f(x_0)$$

$$= \int_{x_0}^{x_1} g(x')dx' \cong g(x_0)\Delta x \qquad (1)$$

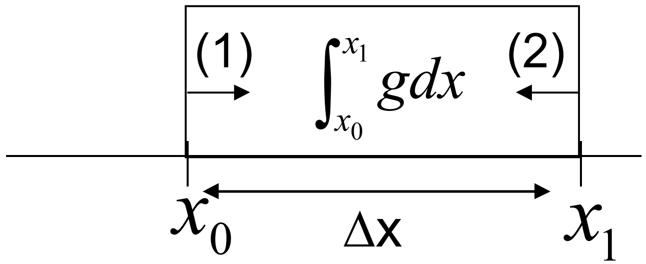
$$\cong \frac{g(x_0) + g(x_1)}{2}(\Delta x) \qquad (2)$$

$$\cong \frac{3g(x_0) - g(x_{-1})}{2}(\Delta x) \qquad (3)$$



Trapezoidal scheme

$$\int g\,dx \cong \int \left(g_i + x g'_i + \frac{1}{2} x^2 g''_i + \frac{1}{6} x^3 g'''_i + \cdots \right) dx$$



1st order approx. estimated error

$$\int_0^{\Delta x} g\,dx \cong g_0 \Delta x \qquad \boxed{+ g'_0 \frac{\Delta x^2}{2}} \qquad + g''_0 \frac{\Delta x^3}{6} \quad + \cdots \qquad (1) \times 1/2$$

$$\int_{-\Delta x}^{0} g\,dx \cong g_1 \Delta x \qquad - g'_1 \frac{\Delta x^2}{2} \qquad + g''_1 \frac{\Delta x^3}{6} \quad + \cdots \qquad (2) \times 1/2$$

$$+ \qquad (g_0 + g_1)\frac{\Delta x}{2} \qquad + \frac{(g'_0 - g'_1)}{2}\frac{\Delta x^2}{2} + \frac{(g''_1 + g''_0)}{2}\frac{\Delta x^3}{6} \cdots$$

2nd order approx.

$$= -\bar{g}'' \frac{\Delta x^3}{4} \qquad \cong \bar{g}'' \frac{\Delta x^3}{6} \qquad \Rightarrow \boxed{-\bar{g}'' \frac{\Delta x^3}{12}}$$

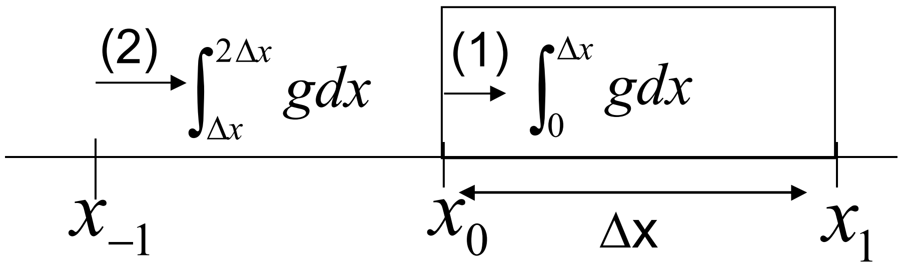
estimated error

$$\bar{g} = \frac{1}{\Delta x} \int_{x_0}^{x_1} g\,dx \cong \frac{g_0 + g_1}{2} \qquad \Rightarrow -\bar{g}'' \frac{\Delta x^2}{12}$$

Error for averaged value

Exercise

Adams－Bashforth scheme



$$\int_0^{\Delta x} g\,dx \cong g_0\Delta x + g'_0\frac{\Delta x^2}{2} + g''_0\frac{\Delta x^3}{6} + \cdots \qquad (1)\ \times\ (3/2)$$

$$+\quad \int_{\Delta x}^{2\Delta x} g\,dx \cong g_{-1}\Delta x + g'_{-1}\frac{3\Delta x^2}{2} + g''_{-1}\frac{8\Delta x^3}{6} + \cdots \qquad (2)\ \times\ (-1/2)$$

$$\left(\frac{3}{2}g_0 - \frac{1}{2}g_{-1}\right)\Delta x + \frac{3(g'_0 - g'_{-1})}{4}\Delta x^2 + \frac{(3g''_0 - 8g''_{-1})}{12}\Delta x^3 \cdots$$

$$\cong \bar{g}''\frac{3\Delta x^3}{4} \qquad \cong -\bar{g}''\frac{5\Delta x^3}{12} \qquad \Rightarrow \bar{g}''\frac{\Delta x^3}{3}$$ Estimated error

Fundamentals for discretizing methods

Finite Volume Method

Flux type differential eq

$$\frac{d}{dx}J(\phi(x)) = g(\phi(x))$$

⇨

Integral eq.

$$J(x_+) - J(x_-) = \int_{x_-}^{x_+} g(x')dx'$$

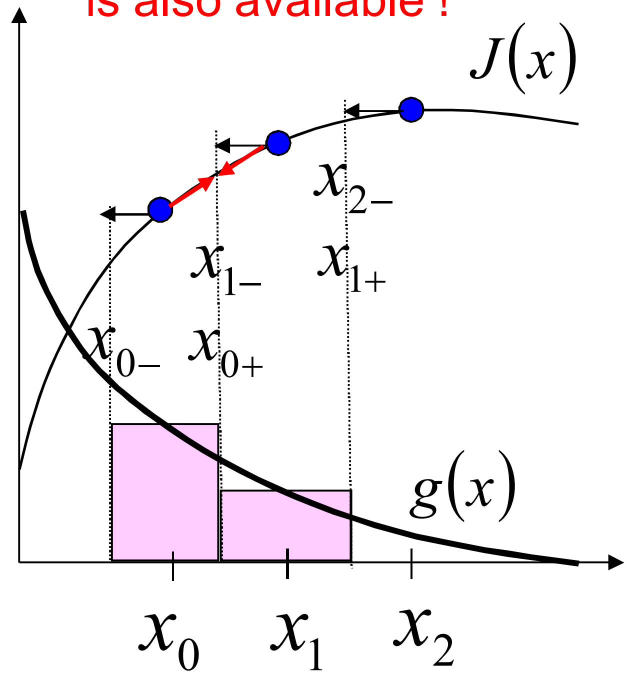
$$J(x_0^-) \cong J(\phi(x_0))$$
$$J(x_1^-)\left[\equiv J(x_0^+)\right] \cong J(\phi(x_1))$$
$$J(x_2^-)\left[\equiv J(x_1^+)\right] \cong J(\phi(x_2))$$
$$\vdots$$

Backward approximation ⇨ Central interpolation is also available !

⇩

$$J(x_{0+}) - J(x_{0-}) \cong g(\phi(x_0))\Delta x$$
$$J(x_{1+}) - J(x_{1-}) \cong g(\phi(x_1))\Delta x$$
$$\vdots$$

Midway-role integration



Finite Volume Method for generalized conservation eq.

Differential form

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot [\mathbf{u}\phi + \mathbf{j}] + S \quad \Rightarrow \quad \nabla \cdot \mathbf{J}(\phi(\mathbf{x})) = g(\phi(\mathbf{x}))$$

$$\mathbf{J}(\phi(\mathbf{x})) \Rightarrow \mathbf{u}\phi + \mathbf{j}(\phi)$$

$$g(\phi(\mathbf{x})) \Rightarrow S(\phi) - \frac{\partial \phi}{\partial t}$$

Integration form

$$\int_\Omega \frac{\partial \phi}{\partial t} dv = \int_\Sigma [\mathbf{u}\phi + \mathbf{j}] \cdot dn + \int_\Omega S dv \quad \Rightarrow \quad J(x_+) - J(x_-) = \int_{x_-}^{x_+} g(x') dx'$$

（1D case）

$$\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial x}[u_x \phi + j_x(\phi)] + S(\phi) \quad \Rightarrow \quad \frac{\partial}{\partial x} J(x, f(x)) = g(x, f(x))$$

$$J(x, \phi(x)) \Rightarrow u_x \phi + j_x(\phi)$$

$$g(x, \phi(x)) \Rightarrow S(\phi) - \frac{\partial \phi}{\partial t}$$

backward
approximation scheme

$$f_0 = f_+ - \left(\frac{\delta x}{2}\right) f'_+ + \frac{1}{2}\left(\frac{\delta x}{2}\right)^2 f''_+ + \cdots \qquad \text{x1/2}$$

$$+ \quad f_1 = f_+ + \left(\frac{\delta x}{2}\right) f'_+ + \frac{1}{2}\left(\frac{\delta x}{2}\right)^2 f''_+ + \cdots \qquad \text{x1/2}$$

$$\frac{f_0 + f_1}{2} = f_+ \quad + \frac{1}{4}\delta x^2 f'' \cdots$$

⋯2nd order accuracy

central interpolation scheme

Estimated error

$$f_0 = f_+ - \left(\frac{\delta x}{2}\right) f'_+ + \frac{1}{2}\left(\frac{\delta x}{2}\right)^2 f''_+ + \cdots \qquad \text{x3/2}$$

$$+ \quad f_{-1} = f_+ - \left(\frac{3\delta x}{2}\right) f'_+ + \frac{1}{2}\left(\frac{3\delta x}{2}\right)^2 f''_+ + \cdots \qquad \text{x-1/2}$$

$$\frac{3f_0 - f_{-1}}{2} = f_+ \quad + \frac{3}{4}\delta x^2 f'' + \cdots$$

⋯2nd order accuracy

extrapolation scheme

Estimated error

Exercise

Estimation of accuracy for Finite Volume Method

Central interpolation

$$J_+ - J_- \cong \frac{1}{2}\left(J_0 + J_1\right) - \frac{1}{2}\left(J_{-1} + J_0\right) + \frac{\Delta x^2}{4}\left(J''_+ - J''_-\right) + \cdots$$

Midway role integration

$$\int_{-\Delta x/2}^{\Delta x/2} g\,dx \cong g\left(f\left(x_0\right)\right)\Delta x + g''_0 \frac{\Delta x^3}{24} + \cdots$$

$$\frac{1}{2}\left(J_0 + J_1\right) - \frac{1}{2}\left(J_{-1} + J_0\right) \cong g\left(f\left(x_0\right)\right)\Delta x \qquad -\frac{\Delta x^3}{4} J''' + \frac{\Delta x^3}{24} g''$$

Discretized eq. Truncation error

For averaged value of volume Δx estimated error in order of $(\Delta x)^2$!!

$$\frac{1}{\Delta x}\int g(\phi(\mathbf{x}))dx \equiv \overline{g(\phi(\mathbf{x}))} = \overline{S(\phi)} - \overline{\frac{\partial \phi}{\partial t}}$$

Computational Fluid Mechanics

Part1：Numerical Methods (3b)

1. Introduction
2. Numerical methods for fluid mechanics (1) ~ governing equations

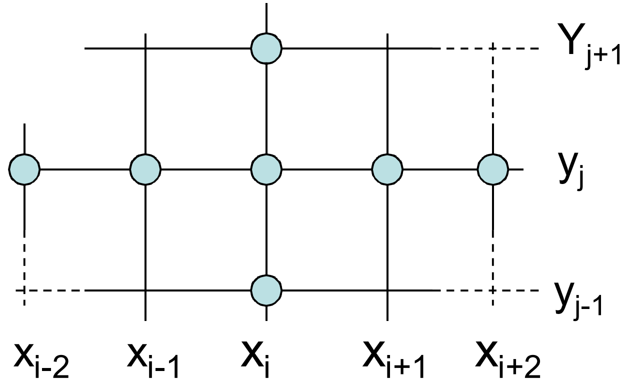
3. Numerical methods for fluid mechanics (2) ~Discretizing schemes
   - Fundamentals for discretizing methods
   - Schemes for convection and diffusion equation
   - Analysis of accuracy and instability

4. Numerical methods for fluid mechanics (3) ~Coupling algorism
5. Numerical methods for fluid mechanics (4) ~Additional problems
6. Reliability of numerical simulation

Schemes for convection and diffusion equation

Finite Deference Method

$$\frac{\partial \phi}{\partial t} = -\mathbf{u}\cdot\nabla\phi - \nabla\cdot\mathbf{j} + S$$



- Convection term

$$u\frac{\partial \phi}{\partial x} \cong \begin{cases} u_i \dfrac{\phi_i - \phi_{i-1}}{\delta x} & (u_i \geq 0) \\ u_i \dfrac{\phi_{i+1} - \phi_i}{\delta x} & (u_i < 0) \end{cases}$$

1st order UPWIND difference scheme

$$v\frac{\partial \phi}{\partial y} \cong v_j \frac{\phi_j - \phi_{j-1}}{\delta y} \quad (v_j \geq 0)$$

$$u\frac{\partial \phi}{\partial x} \cong u_i \frac{\phi_{i+1} - \phi_{i-1}}{2\delta x}$$

2nd order central difference scheme

$$v\frac{\partial \phi}{\partial y} \cong v_j \frac{\phi_{j+1} - \phi_{j-1}}{2\delta y}$$

$$u\frac{\partial \phi}{\partial x} \cong \begin{cases} u_i \dfrac{3\phi_i - 4\phi_{i-1} + \phi_{i-2}}{2\delta x} & (u_i \geq 0) \\ u_i \dfrac{-\phi_{i+1} + 4\phi_i - 3\phi_i}{2\delta x} & (u_i \geq 0) \end{cases}$$

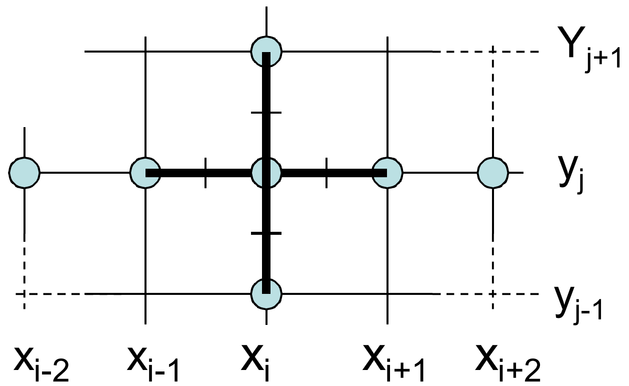
2nd order UPWIND difference scheme

Schemes for convection and diffusion equation

Finite Deference Method (continued)

$$\frac{\partial \phi}{\partial t} = -\mathbf{u}\cdot\nabla\phi - \boxed{\nabla\cdot\mathbf{j}} + S$$

$$\mathbf{j} = -\Gamma\nabla\phi$$



- Diffusion term

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} \cong \frac{J_{i+} - J_{i-}}{\delta x} + \frac{J_{j+} - J_{j-}}{\delta y} \Leftarrow J_{i+} \cong -\Gamma\frac{\phi_{i+1}-\phi_i}{\delta x},\; J_{i-} \cong -\Gamma\frac{\phi_i - \phi_{i-1}}{\delta x}$$

$$\cong \Gamma\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\delta x^2} + \Gamma\frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\delta y^2}$$

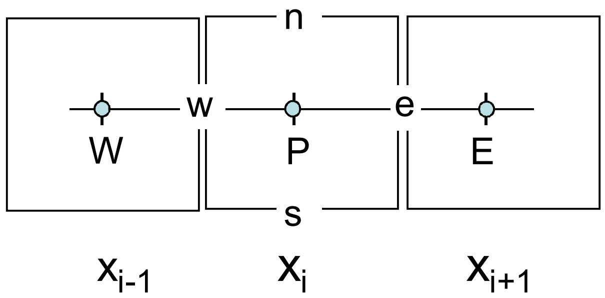
2nd order central difference scheme

$$\Gamma\nabla^2\phi \equiv \Gamma\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \cong \Gamma\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\delta x^2} + \Gamma\frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\delta y^2}$$

Schemes for convection and diffusion equation

Finite Volume (control volume) Method

$$\int\left(\frac{\partial\phi}{\partial t}-S\right)dV=-\int_w(\mathbf{u}\phi+\mathbf{j})\cdot\mathbf{n}dA$$



$$\bar{\phi}\,\delta x\,\delta y\equiv\iint\phi\,dxdy \qquad \bar{J}_w\delta y\equiv\int_w J_x dy$$

$$\left(\frac{\partial\bar{\phi}}{\partial t}-\bar{S}\right)\delta x\,\delta y=-\left\{\left(u_e\bar{\phi}_e+\bar{J}_e\right)-\left(u_w\bar{\phi}_w+\bar{J}_w\right)\right\}\delta y-\left\{\left(u_n\bar{\phi}_n+\bar{J}_n\right)-\left(u_s\bar{\phi}_s+\bar{J}_s\right)\right\}\delta x$$

$$\bar{J}_w\cong-\Gamma\frac{\bar{\phi}_{i+1}-\bar{\phi}_i}{\delta x}$$

Diffusion term ~ 2nd order accuracy

$$\Rightarrow\quad\left(\bar{J}_e-\bar{J}_w\right)dy+\left(\bar{J}_n-\bar{J}_s\right)dx\cong\Gamma\frac{\bar{\phi}_{i+1}-2\bar{\phi}_i+\bar{\phi}_{i-1}}{\delta x^2}+\Gamma\frac{\bar{\phi}_{j+1}-2\bar{\phi}_j+\bar{\phi}_{j-1}}{\delta y^2}$$

$$\overline{\frac{\partial\phi}{\partial t}}\delta x\,\delta y\equiv\iint\left(\frac{\partial\phi}{\partial t}\right)dxdy=\frac{\partial\bar{\phi}}{\partial t}\delta x\,\delta y$$

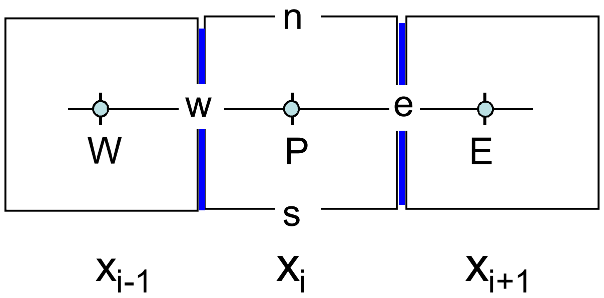
$$\bar{S}\equiv\iint S(\phi)dxdy\cong\bar{S}\left(\bar{\phi}\right)$$

Source term ~ 2nd order accuracy

Schemes for convection and diffusion equation

Finite Volume (control volume) Method

$$\int\left(\frac{\partial\phi}{\partial t}-S\right)dV=-\int_w(\mathbf{u}\phi+\mathbf{j})\cdot\mathbf{n}dA$$



$$\bar{\phi}\,\delta x\delta y\equiv\iint\phi\,dxdy \qquad J_w\equiv\frac{1}{\delta y}\int_w J_x dy$$

$$\left(\frac{\partial\bar{\phi}}{\partial t}-\bar{S}\right)\delta x\delta y=-\{(u_e\phi_e+J_e)-(u_w\phi_w+J_w)\}\delta y-\{(u_n\phi_n+J_n)-(u_s\phi_s+J_s)\}\delta x$$

$$\phi_w\cong\begin{cases}\bar{\phi}_{i-1} & (u_i\ge 0)\\ \bar{\phi}_i & (u_i<0)\end{cases}$$

1st Upwid scheme

Convection term

$$\phi_w=\frac{1}{2}\left(\bar{\phi}_i+\bar{\phi}_{i-1}\right)+\left[-\frac{1}{8}\delta x^2\frac{\partial^2\phi}{\partial x^2}+0\left(\delta x^4\right)\right]$$

2nd Central scheme

$$\phi_w=\frac{1}{8}\left(3\bar{\phi}_i+6\bar{\phi}_{i-1}-\bar{\phi}_{i-2}\right)+\left[\frac{1}{24}\delta x^2\frac{\partial^2 u}{\partial x^2}+0\left(\delta x^3\right)\right]$$

QUICK scheme

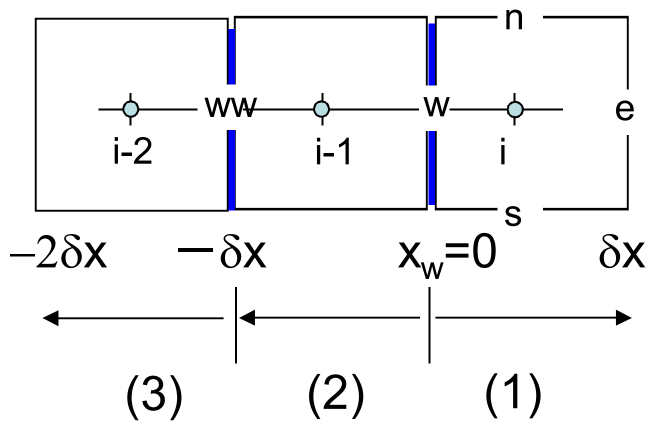
Exercise Analyze the accuracy of FVM schemes by following way

$$\phi = f_w - x f'_w + \frac{x^2}{2} f''_w \cdots$$

(1) $$\bar{\phi}_i \delta x = \int_0^{\delta x} \phi dx = \delta x \phi_w + \frac{\delta x^2}{2} \phi'_w + \frac{\delta x^3}{6} \phi''_w \cdots$$

(2) $$\bar{\phi}_{i-1} \delta x = \int_{-\delta x}^{0} \phi dx = \delta x \phi_w - \frac{\delta x^2}{2} \phi'_w + \frac{\delta x^3}{6} \phi''_w \cdots$$

(3) $$\bar{\phi}_{i-2} \delta x = \int_{-2\delta x}^{-\delta x} \phi dx = \delta x \phi_w - \frac{3\delta x^2}{2} \phi'_w + \frac{7\delta x^3}{6} \phi''_w \cdots$$

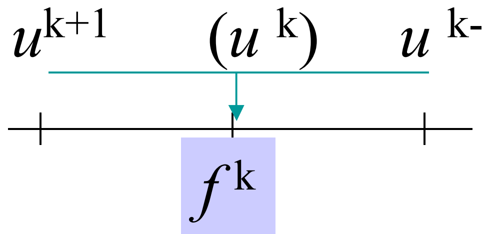


Time marching schemes

$$\frac{\partial u}{\partial t} = f \quad \Rightarrow \quad \begin{cases} f \Rightarrow f^{k} & \text{(explicit scheme)} \\ f \Rightarrow f^{k+1} & \text{(implicit scheme)} \end{cases}$$

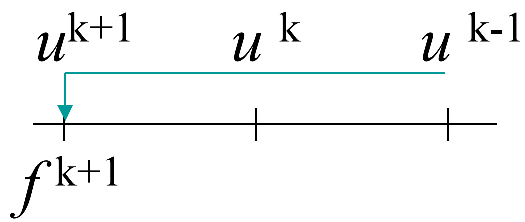
- (2nd order) leap-frog

$$\frac{u^{k+1} - u^{k-1}}{2\delta t} = f^{k}$$



- 2nd order backward scheme

$$\frac{3u^{k+1} - 4u^{k} + u^{k-1}}{2\delta t} = f^{k+1}$$



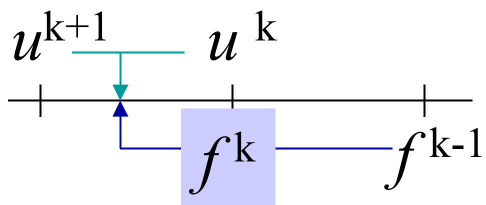
Time marching schemes

$$\frac{\partial u}{\partial t} = f \quad \Rightarrow \quad \frac{u^{k+1} - u^k}{\delta t} = \bar{f}$$

•mid-point rule

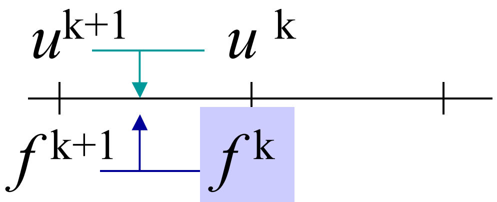
- 2nd order Adams-Bashforth

$$\frac{u^{k+1} - u^k}{\delta t} = \frac{1}{2}\left(3f^k - f^{k-1}\right)$$



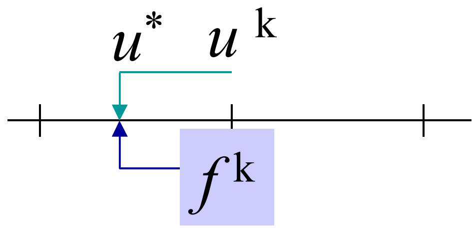
- (2nd order) Crank-Nicolson

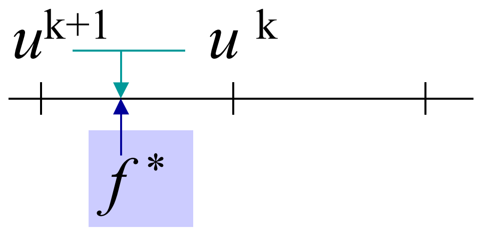
$$\frac{u^{k+1} - u^k}{\delta t} = \frac{1}{2}\left(f^{k+1} + f^k\right)$$



- predictor (Euler) -corrector (Crank-Nicolson)

$$\frac{u^* - u^k}{\delta t} = f\left(u^k\right) \Rightarrow \frac{u^{k+1} - u^k}{\delta t} = \hat{f}\left(u^*\right)$$



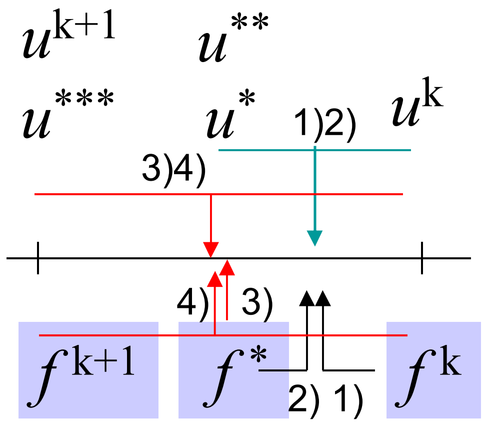


- 4th order Runge-Kutta

1) $u^* = u^k + \frac{\delta t}{2} f\left(u^k\right)$

2) $u^{**} = u^k + \frac{\delta t}{2} f^*\left(u^*\right)$

3) $u^{***} = u^k + \delta t \cdot f^*\left(u^{**}\right)$



4) $u^{k+1} = u^k + \frac{\delta t}{6}\left\{f^k\left(u^k\right) + 2f^*\left(u^*\right) + 2f^*\left(u^{**}\right) + f^{k+1}\left(u^{***}\right)\right\}$

Computational Fluid Mechanics

Part1：Numerical Methods (3c)

1. Introduction
2. Numerical methods for fluid mechanics (1) ~ governing equations

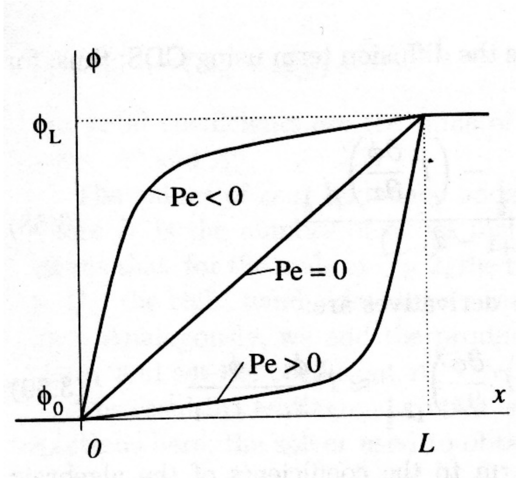
3. Numerical methods for fluid mechanics (2) ~Discretizing schemes
   - Fundamentals for discretizing methods
   - Schemes for convection and diffusion equation
   - Analysis of accuracy and instability

4. Numerical methods for fluid mechanics (3) ~Coupling algorism
5. Numerical methods for fluid mechanics (4) ~Additional problems
6. Reliability of numerical simulation

Numerical accuracy

- Steady C-D problem

$$c\frac{\partial u}{\partial x} = \Gamma\frac{\partial^2 u}{\partial x^2} \qquad Pe = \frac{c\delta x}{\Gamma}$$

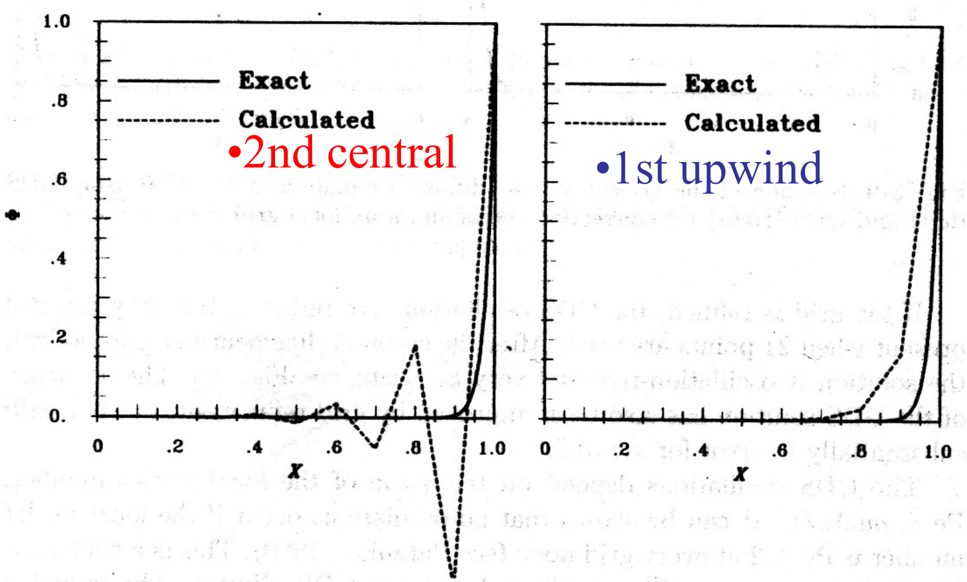


•2nd central

$$c\frac{u_{i+1} - u_{i-1}}{2\delta x} = \Gamma\frac{u_{i+1} - 2u_i + u_{i-1}}{\delta x^2}$$

•1st upwind

$$c\frac{u_i - u_{i-1}}{\delta x} = \Gamma\frac{u_{i+1} - 2u_i + u_{i-1}}{\delta x^2}$$



Numerical accuracy

- Comparison between 1st Upwind and 2nd central schemes

$$\frac{\partial u}{\partial x} = \frac{1}{\delta x}(u_i - u_{i-1}) + \left[\frac{\delta x}{2}\frac{\partial^2 u}{\partial x^2} + 0(\delta x^2)\right]$$

1st upwind

⇩

$$= \frac{1}{\delta x}(u_i - u_{i-1}) + \frac{1}{2\delta x}(u_{i+1} - 2u_i + u_{i-1}) + 0(\delta x^2)$$

2nd central

$$\therefore \frac{\partial u}{\partial t} = (1^{st}\ \text{upwind}) - |u|\frac{\delta x}{2}\frac{\partial^2 u}{\partial x^2}$$

$=\Gamma_{eff}$

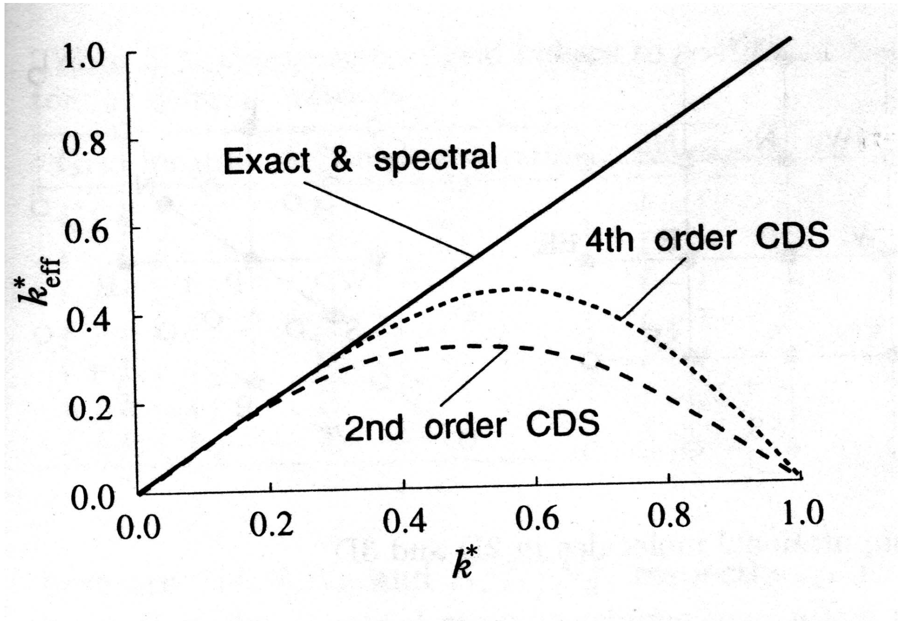
Upwind＝Numerical diffusion

Numerical accuracy

- Fourier expansion

$$u = \sum a_k e^{ikx}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \sum a_k \frac{\partial e^{ikx}}{\partial x} = \sum a_k ike^{ikx}$$



- effective wave number

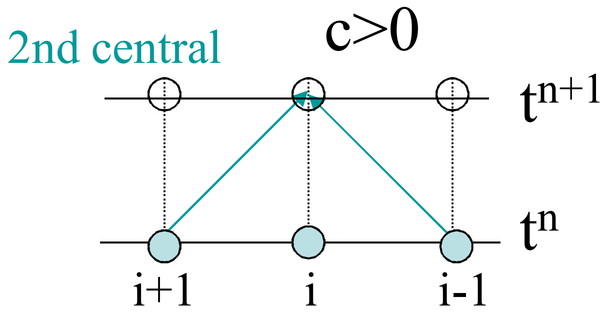
$$\frac{\partial e^{ikx}}{\partial x} \approx \frac{e^{ik(x+\delta x)} - e^{ik(x-\delta x)}}{2\delta x} = i\boxed{\frac{\sin(k\delta x)}{\delta x}}e^{ikx}$$

$$\equiv k_{eff} \approx k - \frac{k^3 \delta x^2}{6}$$

Numerical instability

- von Neuman's condition

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \quad (\text{i.c.} \quad u_{(t_n)} = e^{ikx})$$



$$\frac{u_i^{n+1} - u_i^n}{\delta t} = -c\frac{u_{i+1}^n - u_{i-1}^n}{2\delta x}$$

$$u_i^{n+1} = \left\{1 - i\frac{c\delta t}{\delta x}\sin(k\delta x)\right\}u^n$$

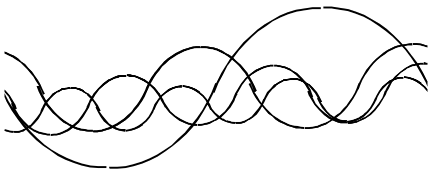
$$|\cdots| = \sqrt{1 + c^2\lambda^2 \sin^2(k\delta x)}$$

$$\max|\cdots| = \sqrt{1 + c^2\lambda^2} > 1$$

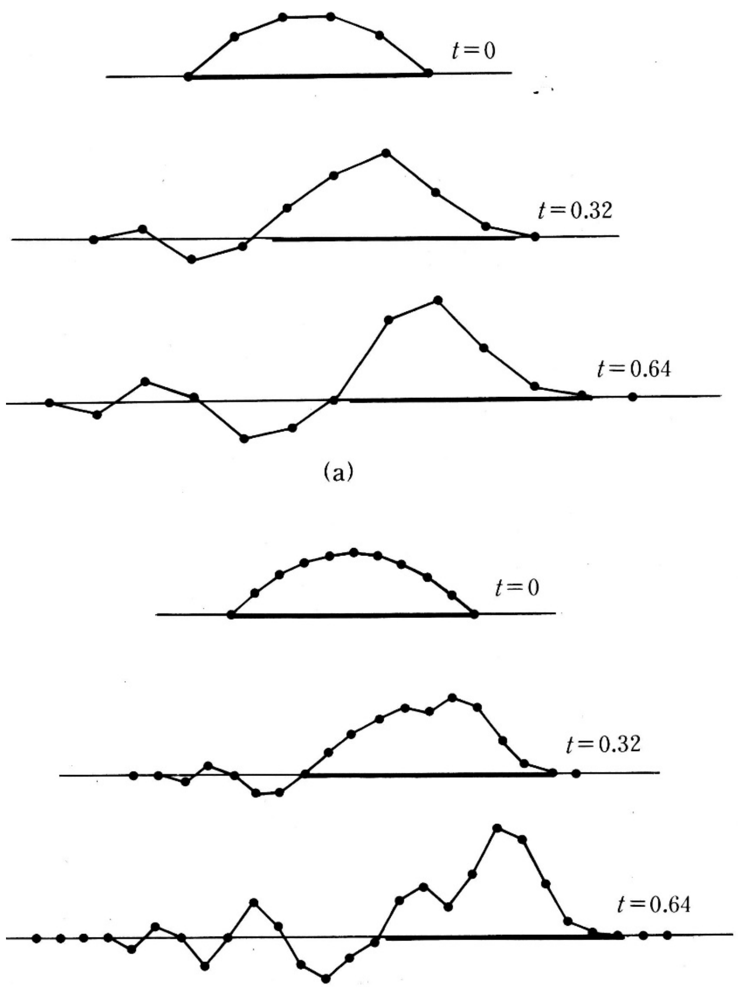
$$\left(if \ \frac{\delta t}{\delta x} = \lambda \ (\text{const.})\right)$$

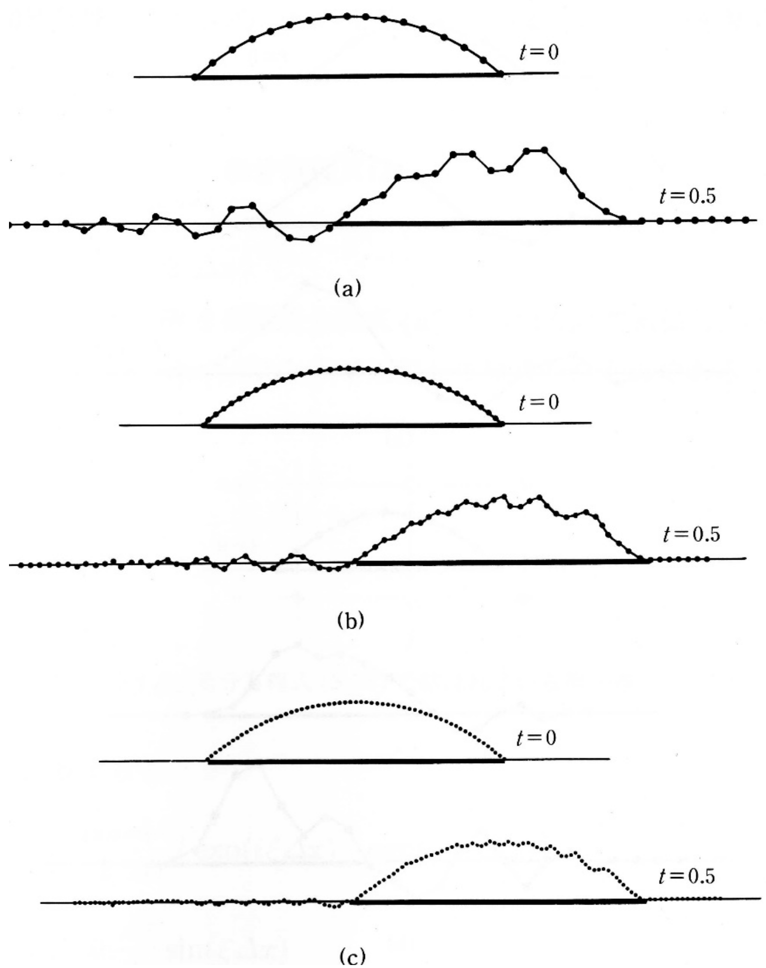
$$\max|\cdots| = \sqrt{1 + c^2\rho\delta t} \le 1 + \frac{c^2\rho}{2}\delta t$$

$$\left(if \ \frac{\delta t}{\delta x^2} = \rho \ (\text{const.})\right)$$



$$u_{(t_n)} = e^{ikx}$$


t=0
t=0.32
t=0.64
(a)
t=0
t=0.32
t=0.64


t=0
t=0.5
(a)
t=0
t=0.5
(b)
t=0
t=0.5
(c)

Numerical instability

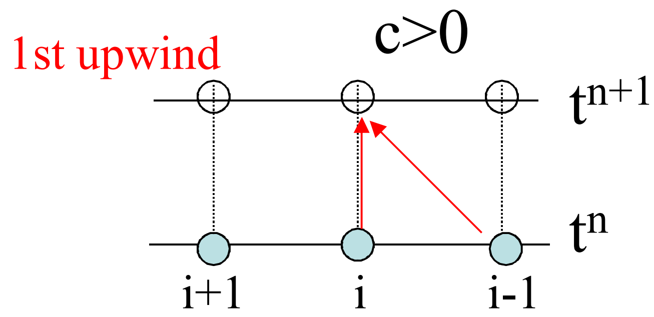
- von Neuman's condition

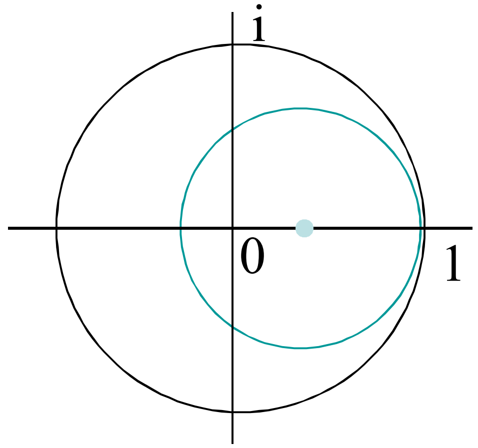
$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \quad (\text{i.c.} \quad u_{(t_n)} = e^{ikx})$$

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = -c\frac{u_i^n - u_{i-1}^n}{\delta x}$$

$$u_i^{n+1} = \left\{\left(1 - \frac{c\delta t}{\delta x}\right) - \frac{c\delta t}{\delta x} e^{-ik\delta x}\right\} u^n$$

CFL condition $c\frac{\delta t}{\delta x} \leq 1$





Exercise

Neumann’s condition of diffusion step

$$\frac{\partial \phi}{\partial t} = \Gamma \frac{\partial^2 \phi}{\partial x^2} \quad \Rightarrow \quad \phi = \phi_n \exp\left(inx - \frac{t}{\Gamma} \right)$$

- Euler (1^{st} explicit)

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \Gamma \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\delta x^2} \qquad \text{... stable ?}$$

- Crank-Ncolsin (2^{nd} implicit)

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \Gamma \frac{\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{2\delta x^2} \qquad ?$$