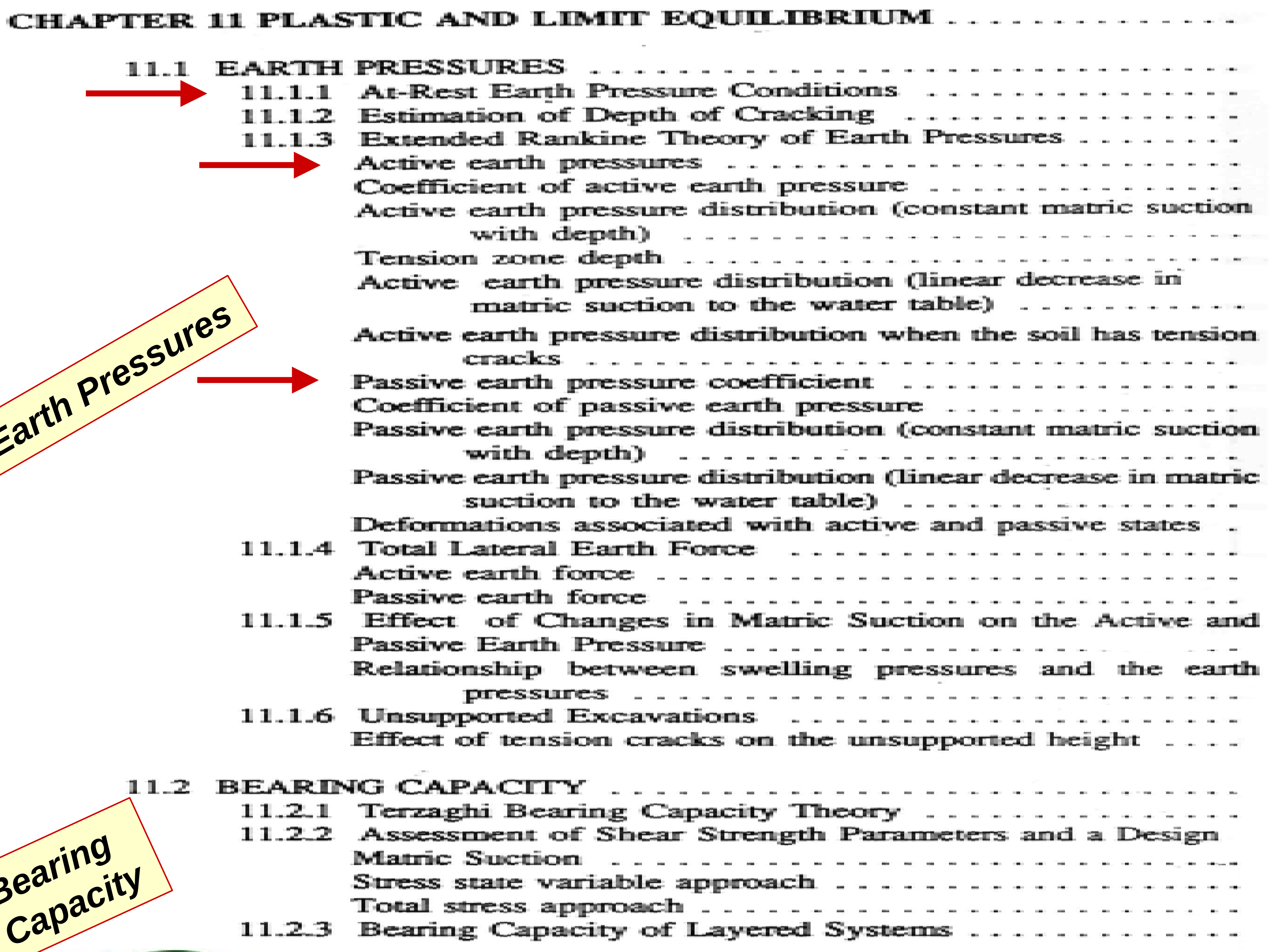
# Unsaturated Soil Technology



## CHAPTER 11 PLASTIC AND LIMIT EQUILIBRIUM

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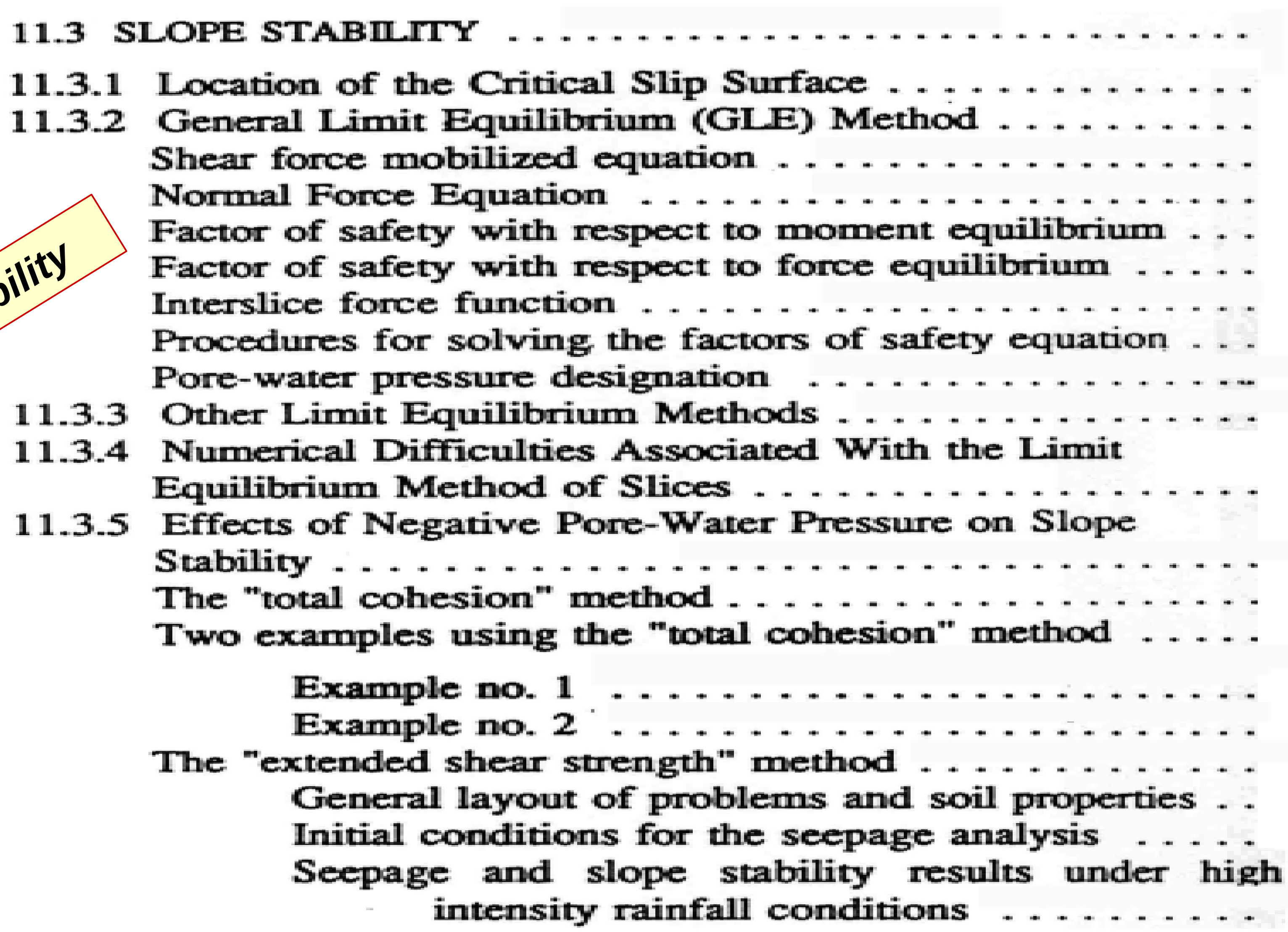
Earth Pressures

Bearing Capacity

**Slope Stability**



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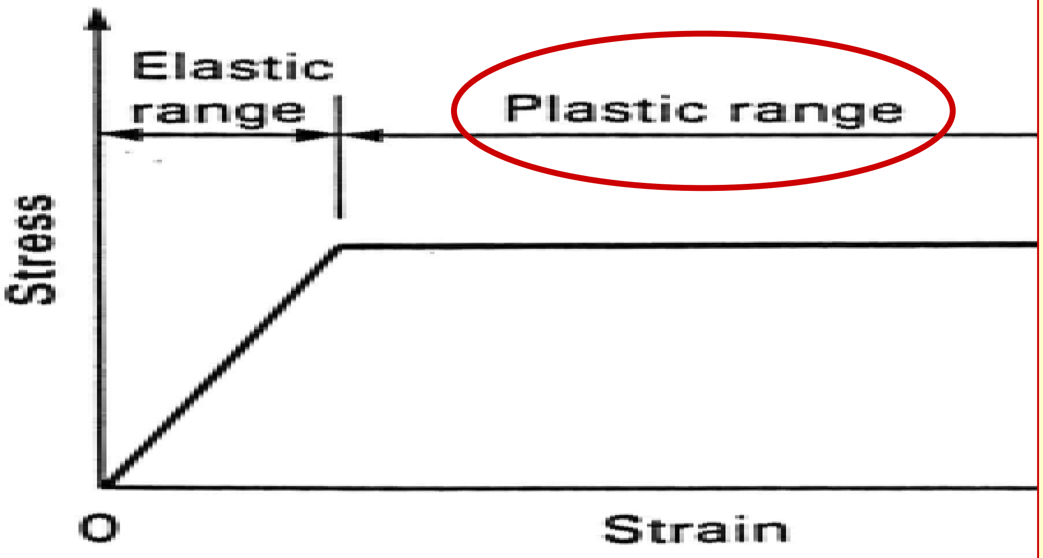
# CHAPTER 11
# PLASTIC AND LIMIT EQUILIBRIUM



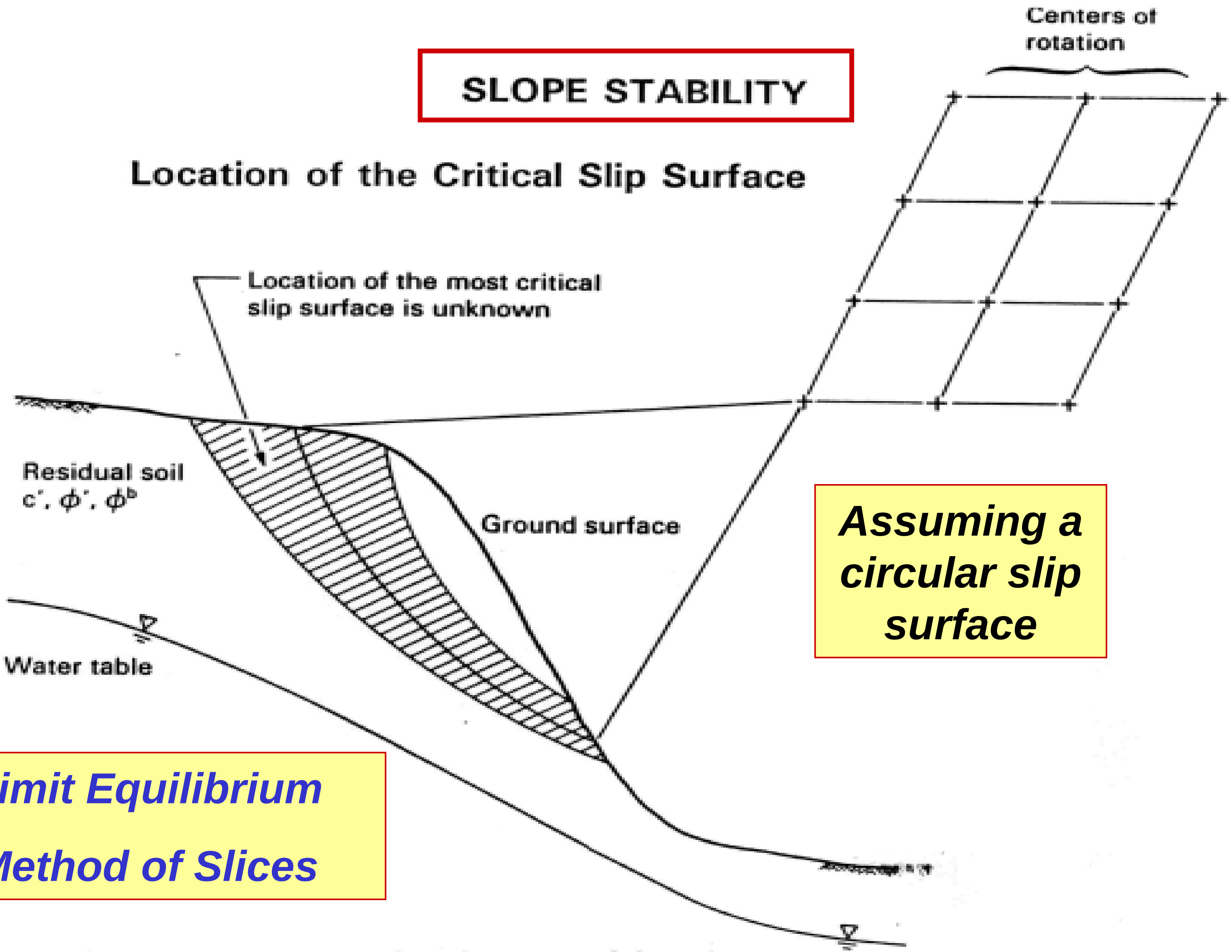
*Three categories of problems typically analyzed using a description of failure conditions; namely*

1. *Earth pressures*
2. *Bearing capacity*
3. *Slope stability*

Idealized elastic-plastic behavior giving rise to two categories of deformation analyses

# SLOPE STABILITY

## Location of the Critical Slip Surface



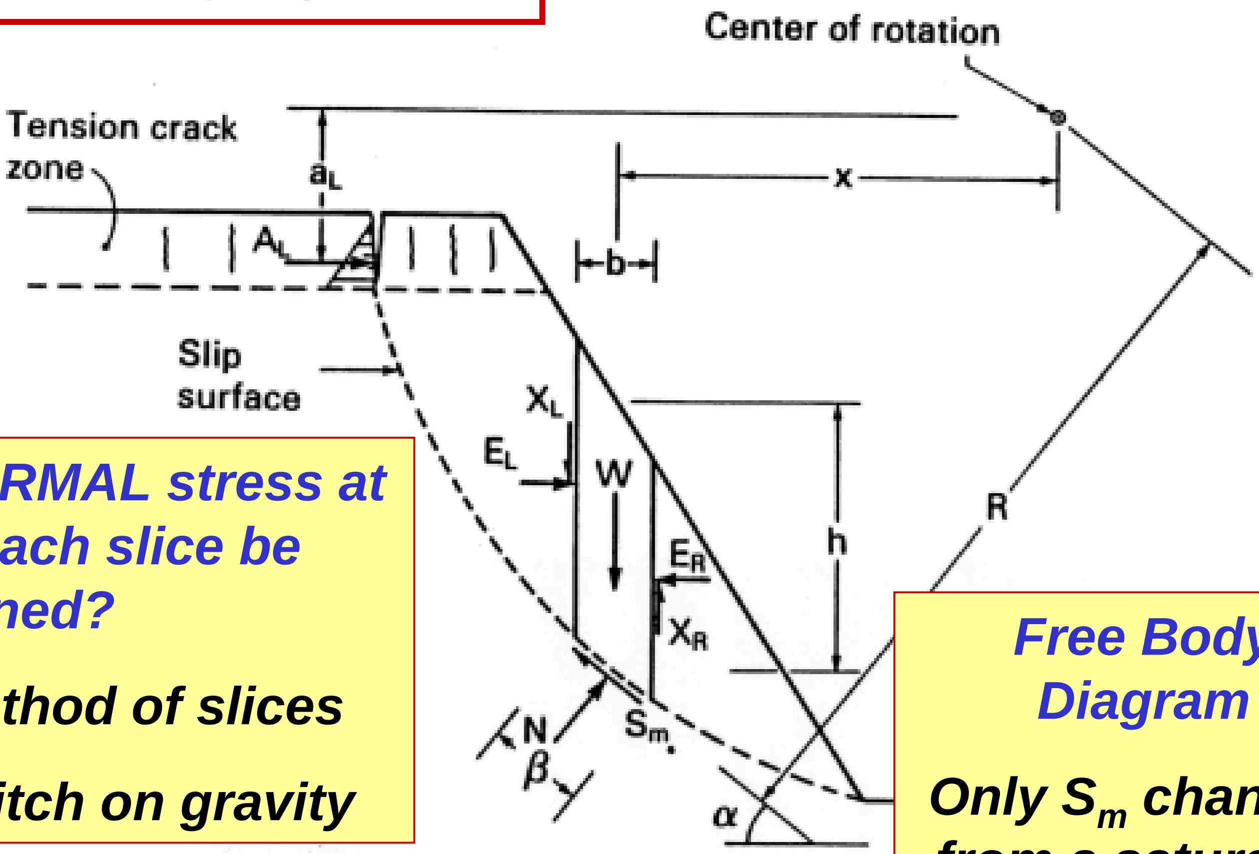
***Assuming a circular slip surface***

***Limit Equilibrium***

***Method of Slices***

A steep natural slope with deep groundwater table

# General Limit Equilibrium (GLE) Method



***How can the NORMAL stress at the base of each slice be obtained?***

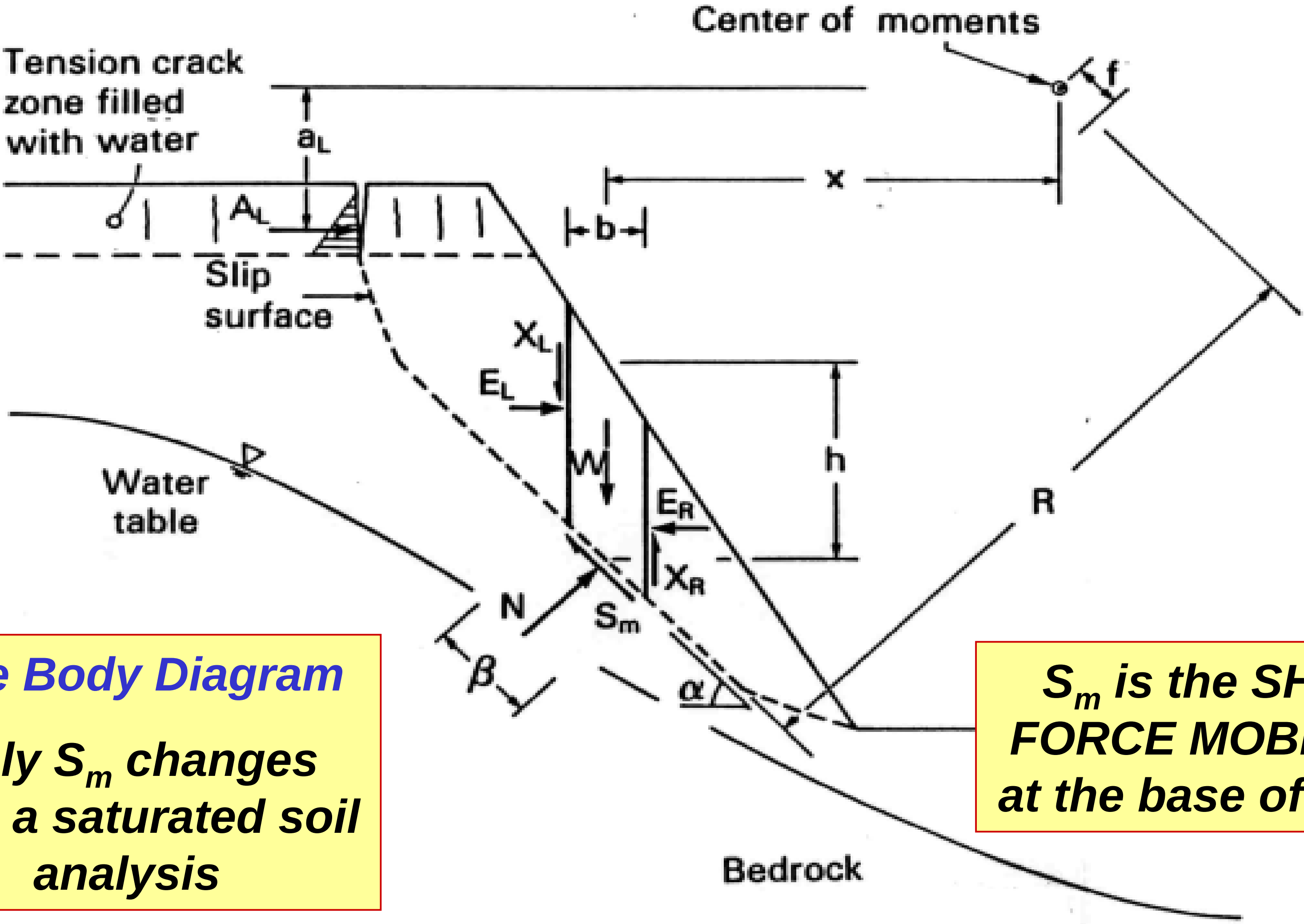
***Method #1: Method of slices***

***Methos #2: Switch on gravity***

***Free Body Diagram***

***Only $S_m$ changes from a saturated soil analysis***

Forces acting on a slice through a sliding mass with a circular slip surface



***Free Body Diagram***

***Only $S_m$ changes from a saturated soil analysis***

***$S_m$ is the SHEAR FORCE MOBILIZED at the base of a slice***

Forces acting on a slice through a sliding mass with a composite slip surface

# Shear force mobilized equation

$$S_m = \frac{\beta}{F}\{c' + (\sigma_n - u_a)\tan\phi' + (u_a - u_w)\tan\phi^b\}$$

***Approach #1***

where:

$\sigma_n$ = total stress normal to the base of a slice

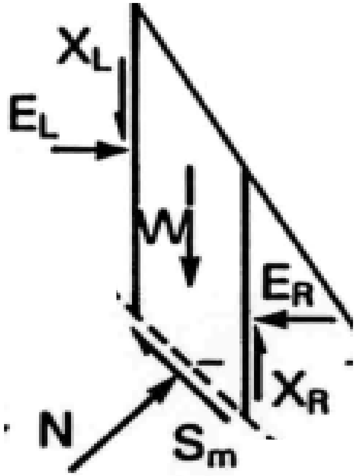
$F$ = factor of safety which is defined as a factor by which the shear strength parameters must be reduced in order to bring the soil mass into a state of limiting equilibrium along the assumed slip surface

$$S_m = \frac{\beta}{F}\{c + (\sigma_n - u_a)\tan\phi'\}$$

***Approach #2***

where:

$c$ = total cohesion of the soil which has two components (i.e., $c' + (u_a - u_w)\tan\phi^b$)



## Normal Force Equation

$$W - (X_R - X_L) - S_m \sin\alpha - N\cos\alpha = 0$$

## *Summation of Forces in the Vertical Direction*

$$W - (X_R - X_L) - \left\{\frac{c'\beta}{F} + \frac{N\tan\phi'\beta}{F} - \frac{u_a\tan\phi'\beta}{F} + \frac{(u_a - u_w)\tan\phi^b\beta}{F}\right\}\sin\alpha - N\cos\alpha = 0$$

or,

$$N\left(\cos\alpha + \frac{\sin\alpha\tan\phi'}{F}\right) = W - (X_R - X_L) - \frac{c'\beta\sin\alpha}{F} + u_a\frac{\beta\sin\alpha}{F}(\tan\phi' - \tan\phi^b) + u_w\frac{\beta\sin\alpha}{F}\tan\phi^b$$

# *Rearranging and solving for the* ***Normal Stress*** *at the base of a slice*

$$N = \frac{W - (X_R - X_L) - \dfrac{c'\beta\sin\alpha}{F} + u_a\dfrac{\beta\sin\alpha}{F}(\tan\phi' - \tan\phi^b) + u_w\dfrac{\beta\sin\alpha}{F}\tan\phi^b}{m_\alpha}$$

where:

$$m_\alpha = \cos\alpha + (\sin\alpha\tan\phi')/F$$

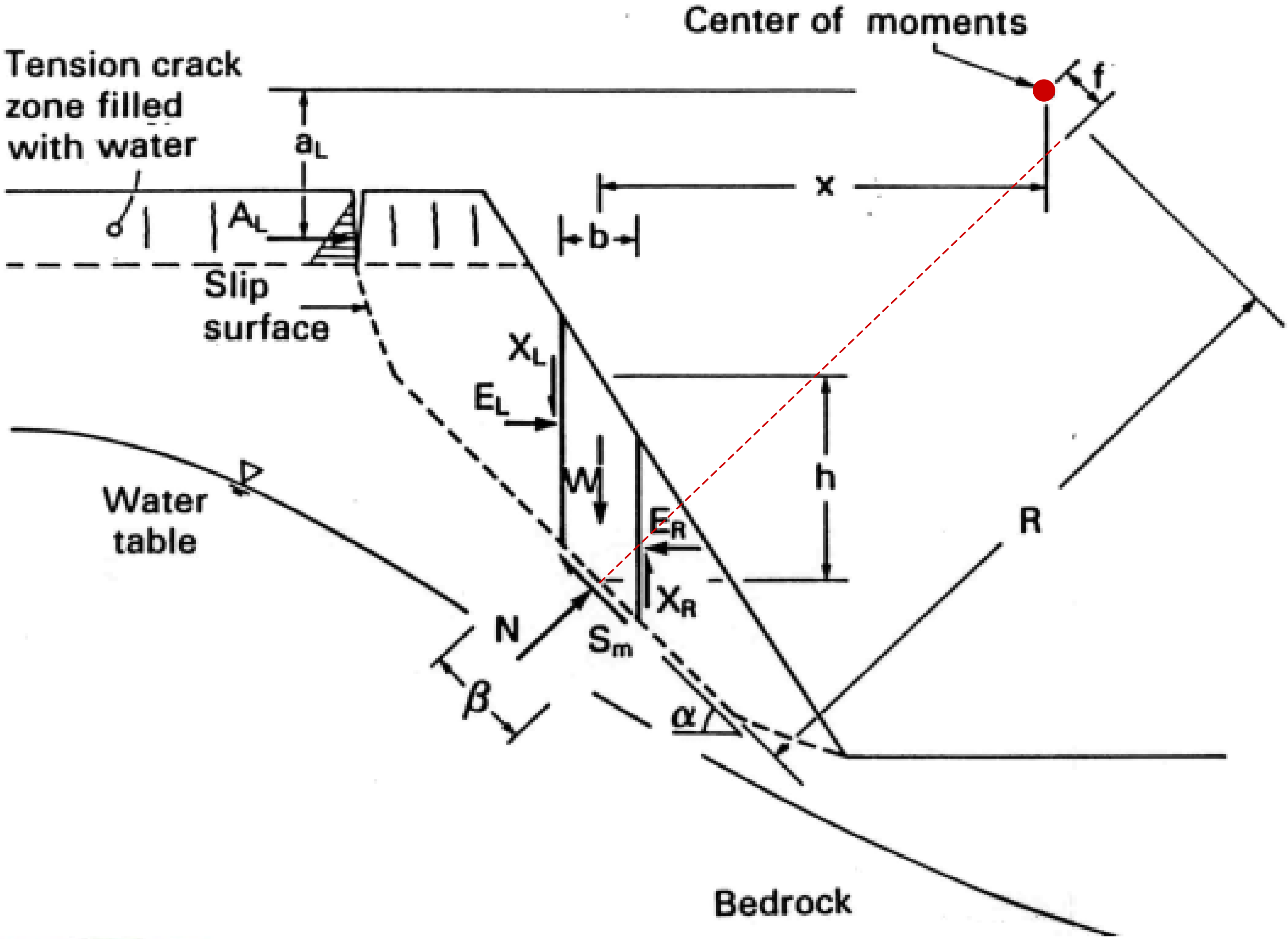
**Notes:**

***Similar for saturated and unsaturated soils except tan $\phi'$ becomes tan $\phi^b$ for an unsaturated soil***

$$\boxed{N = \frac{W - (X_R - X_L) - \dfrac{c'\beta\sin\alpha}{F} + u_w\dfrac{\beta\sin\alpha}{F}\tan\phi^b}{m_\alpha}}$$

***F in N makes $F_m$ and $F_f$ nonlinear***

# *Summation of Moments About a Common Point*



# Factor of safety with respect to moment equilibrium

$$A_L a_L + \sum W x - \sum N f - \sum S_m R = 0$$

$$A_L a_L + \sum W x - \sum N f = \frac{1}{F_m} \sum [c' \beta R + \{N \tan\phi' - u_a \tan\phi' \beta + u_a - u_w) \tan\phi^b \beta\} R]$$

where:

$F_m$ = factor of safety with respect to moment equilibrium

$$F_m = \frac{\sum [c' \beta R + \{N - u_w \beta \frac{\tan\phi^b}{\tan\phi'} - u_a \beta (1 - \frac{\tan\phi^b}{\tan\phi'})\} R \tan\phi']}{A_L a_L + \sum W x - \sum N f}$$

$$F_m = \frac{\sum [c' \beta R + \{N - u_w \beta \frac{\tan\phi^b}{\tan\phi'}\} R \tan\phi']}{A_L a_L + \sum W x - \sum N f}$$

***F in N makes $F_m$ nonlinear***

$$\boxed{F_m = \frac{\sum [c' \beta + \{N - u_w \beta - u_a \beta\} \tan\phi'] R}{A_L a_L + \sum W x}}$$

***Summation of Moments about a Common Axis***

# Factor of safety with respect to force equilibrium

$$\frac{1}{F_f}\sum [c'\beta\cos\alpha + \{N\tan\phi' - u_a\tan\phi'\beta + (u_a - u_w)\tan\phi^b\beta\}] = A_L + \sum N\sin\alpha$$

***Summation of Forces in the Horizontal Direction***

where:

$F_f$ = factor of safety with respect to force equilibrium

$$F_f = \frac{\sum [c'\beta\cos\alpha + \{N - u_w\beta\frac{\tan\phi^b}{\tan\phi'} - u_a\beta(1 - \frac{\tan\phi^b}{\tan\phi'})\}\tan\phi'\cos\alpha]}{A_L + \sum N\sin\alpha}$$

$$\boxed{F_f = \frac{\sum [c'\beta\cos\alpha + \{N - u_w\beta\frac{\tan\phi^b}{\tan\phi'}\}\tan\phi'\cos\alpha]}{A_L + \sum N\sin\alpha}}$$

***F in N makes $F_f$ nonlinear***

# *Designation of an Inter-slice Force Function*

$$X = \lambda f(x) E$$

where:

$f(x)$ = a functional relationship which describes the manner in which the magnitude of X / E varies across the slip surface

$\lambda$ = a scaling constant which represents the percentage of the function, f(x), used for solving the factor of safety equations

$$f(x) = Ke^{-(C^n \omega^n)/2}$$

***Form of an Extended Error Function***

where:

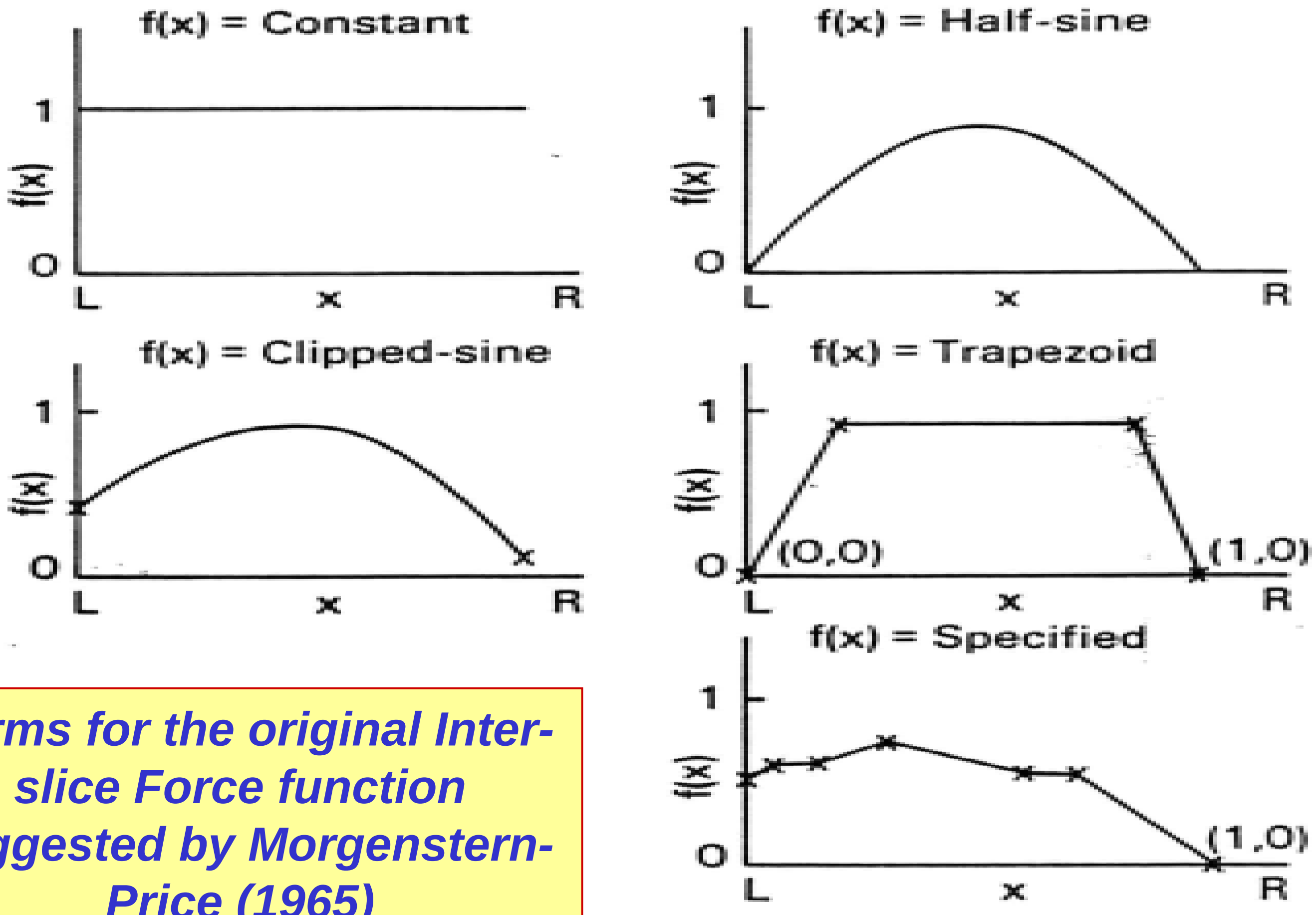
e = base of the natural logarithm

K = magnitude of the interslice force function at midslope (i.e., maximum value)

C = variable to define the inflection points

n = variable to specify the flatness or sharpness of curvature

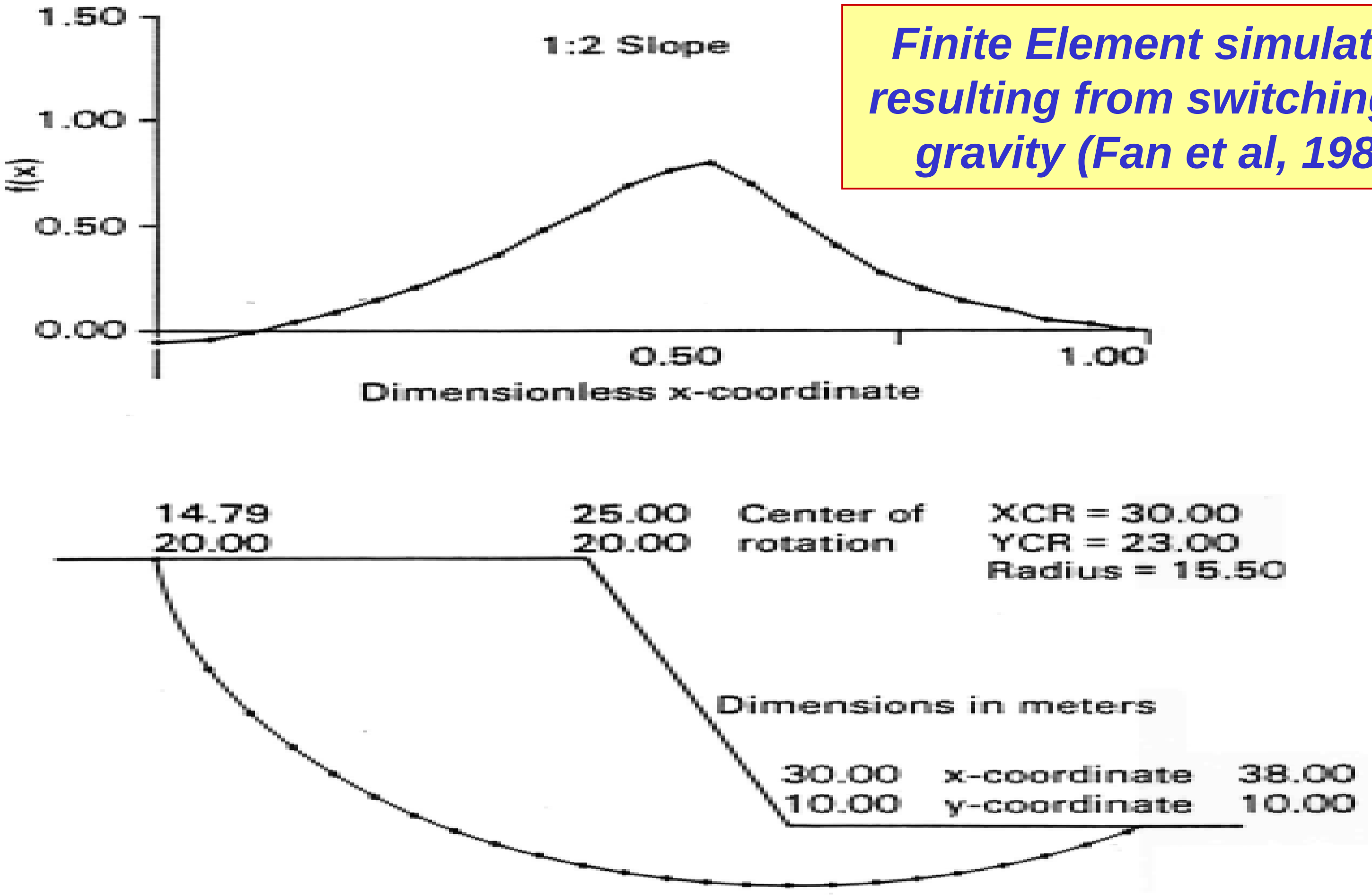
$\omega$ = dimensionless x-position relative to the midpoint of the slope



***Forms for the original Inter-slice Force function suggested by Morgenstern-Price (1965)***

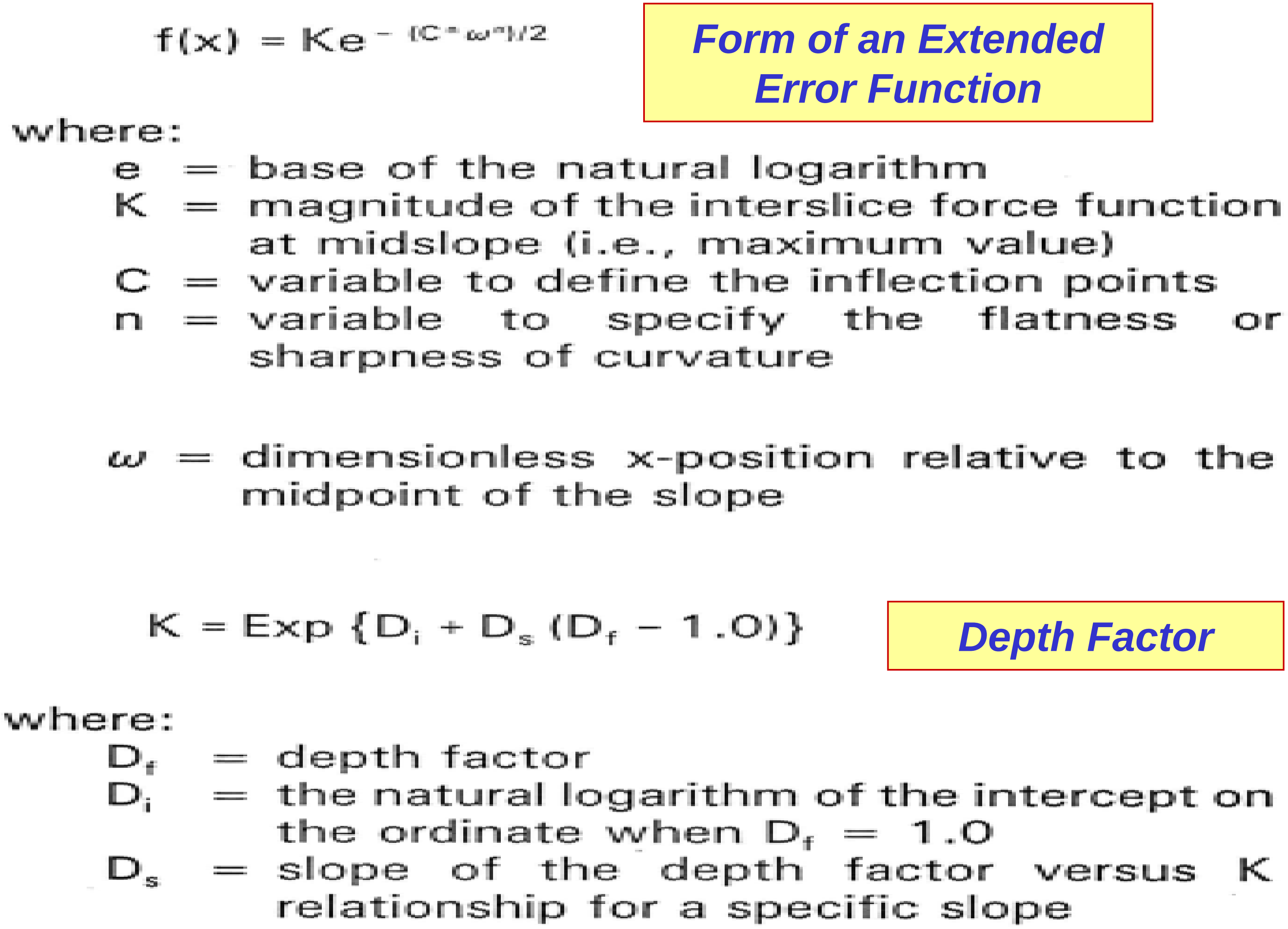
Various possible interslice force functions

***Finite Element simulation resulting from switching on gravity (Fan et al, 1986)***



The interslice force function for a deep-seated slip surface through a one horizontal to two vertical slope

# *Designation of an Inter-slice Force Function*



$$f(x) = Ke^{-(C^n \omega^n)/2}$$

***Form of an Extended Error Function***

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e = base of the natural logarithm

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C = variable to define the inflection points

n = variable to specify the flatness or sharpness of curvature

$\omega$ = dimensionless x-position relative to the midpoint of the slope

$$K = \mathrm{Exp}\,\{D_i + D_s\,(D_f - 1.0)\}$$

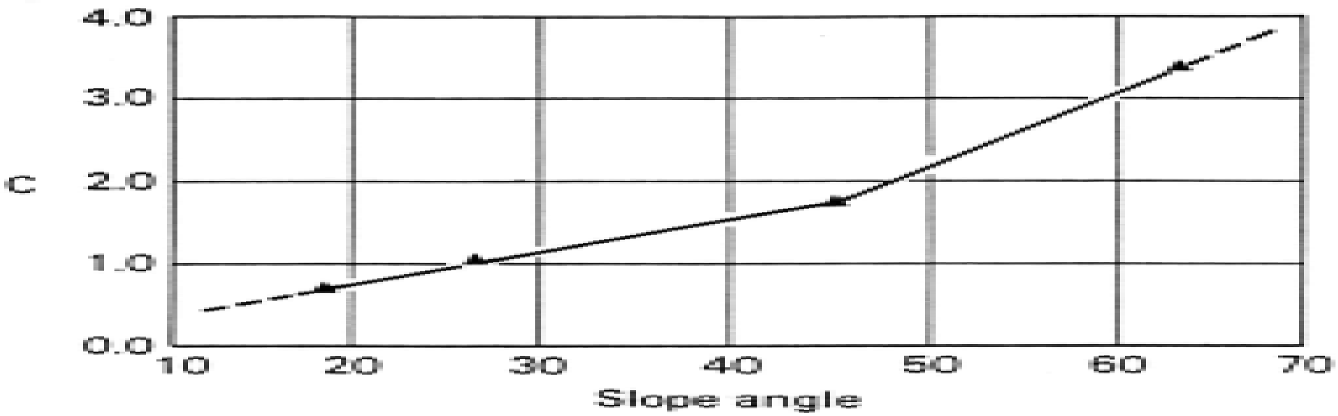
***Depth Factor***

where:

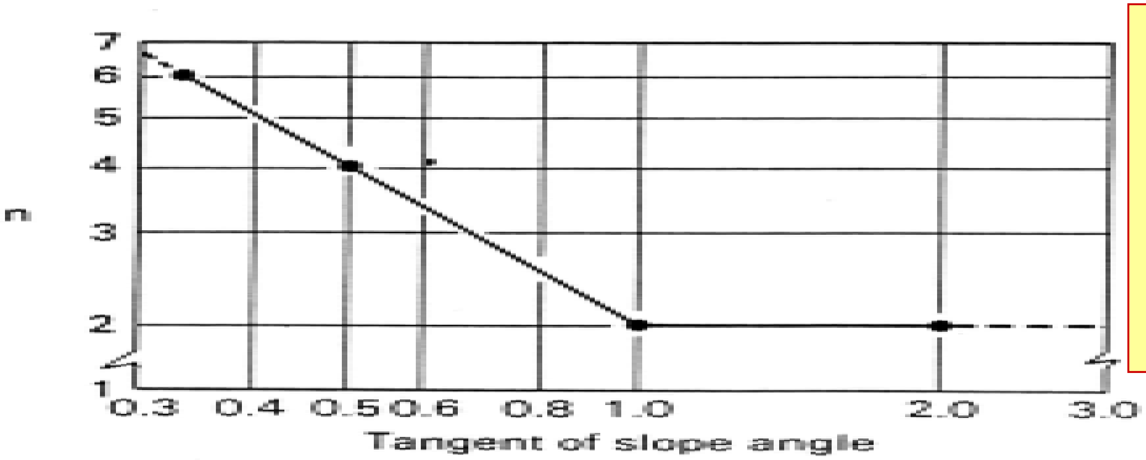
$D_f$ = depth factor

$D_i$ = the natural logarithm of the intercept on the ordinate when $D_f$ = 1.0

$D_s$ = slope of the depth factor versus K relationship for a specific slope



(a) C values versus the slope angle

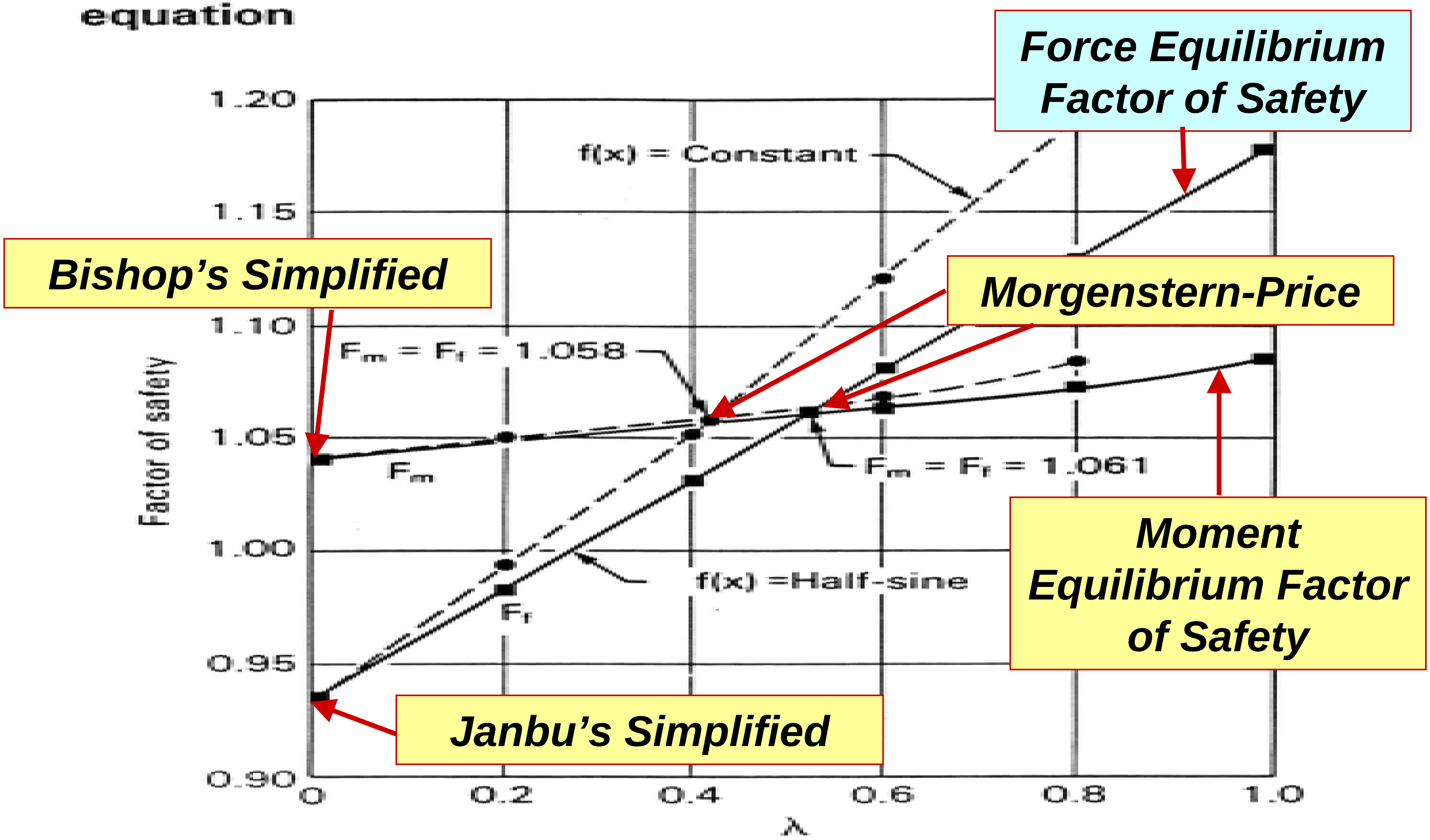


(b) n values versus the tangent of the slope angle

***All variables for the Inter-slice Force function can be estimated from the slope angle***

Values of 'C' and 'n' coefficients versus the slope angle

# Procedures for solving the factors of safety equation



*Force Equilibrium Factor of Safety*

*Bishop's Simplified*

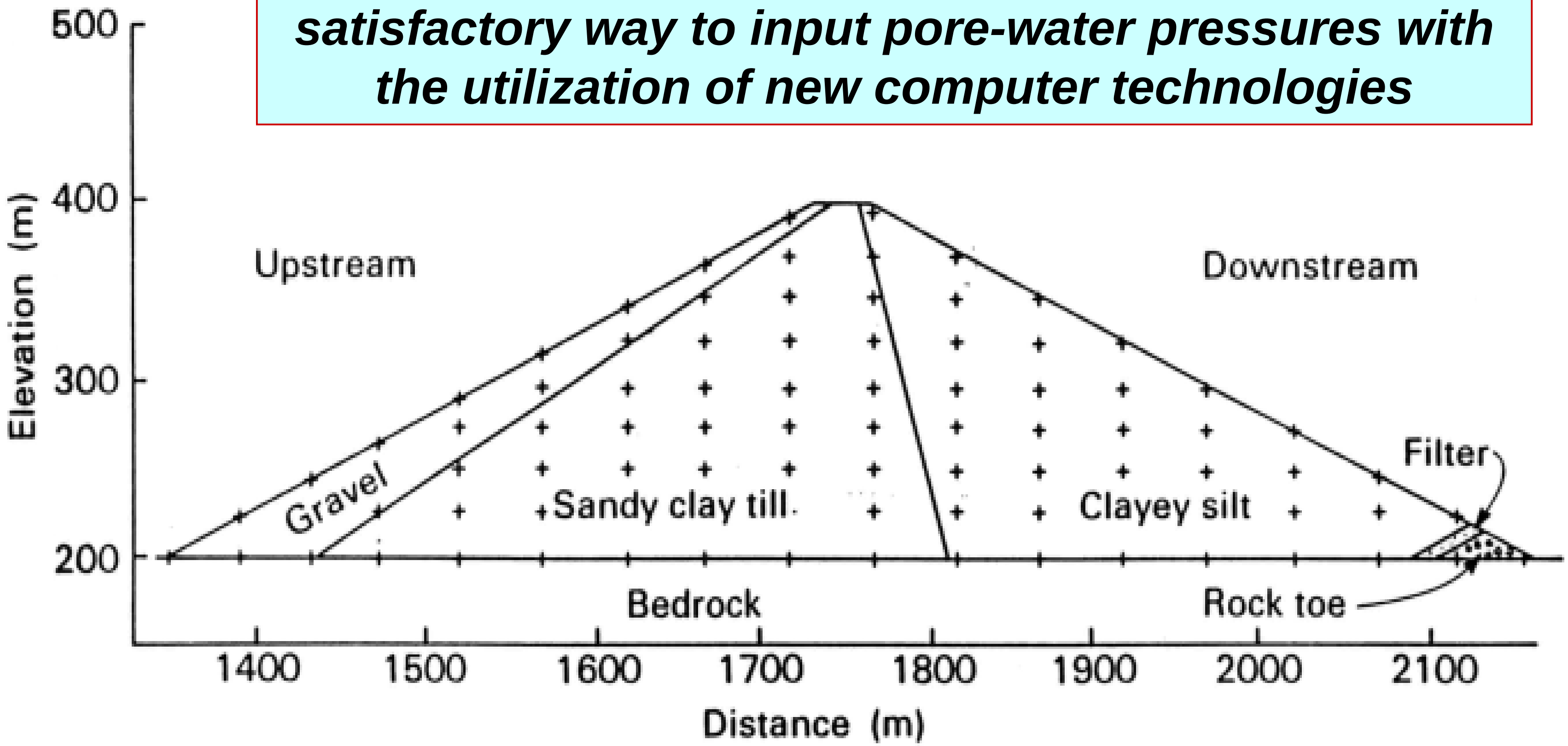
*Morgenstern-Price*

*Moment Equilibrium Factor of Safety*

*Janbu's Simplified*

Variation of moment and force equilibrium factors of safety with respect to lambda, λ

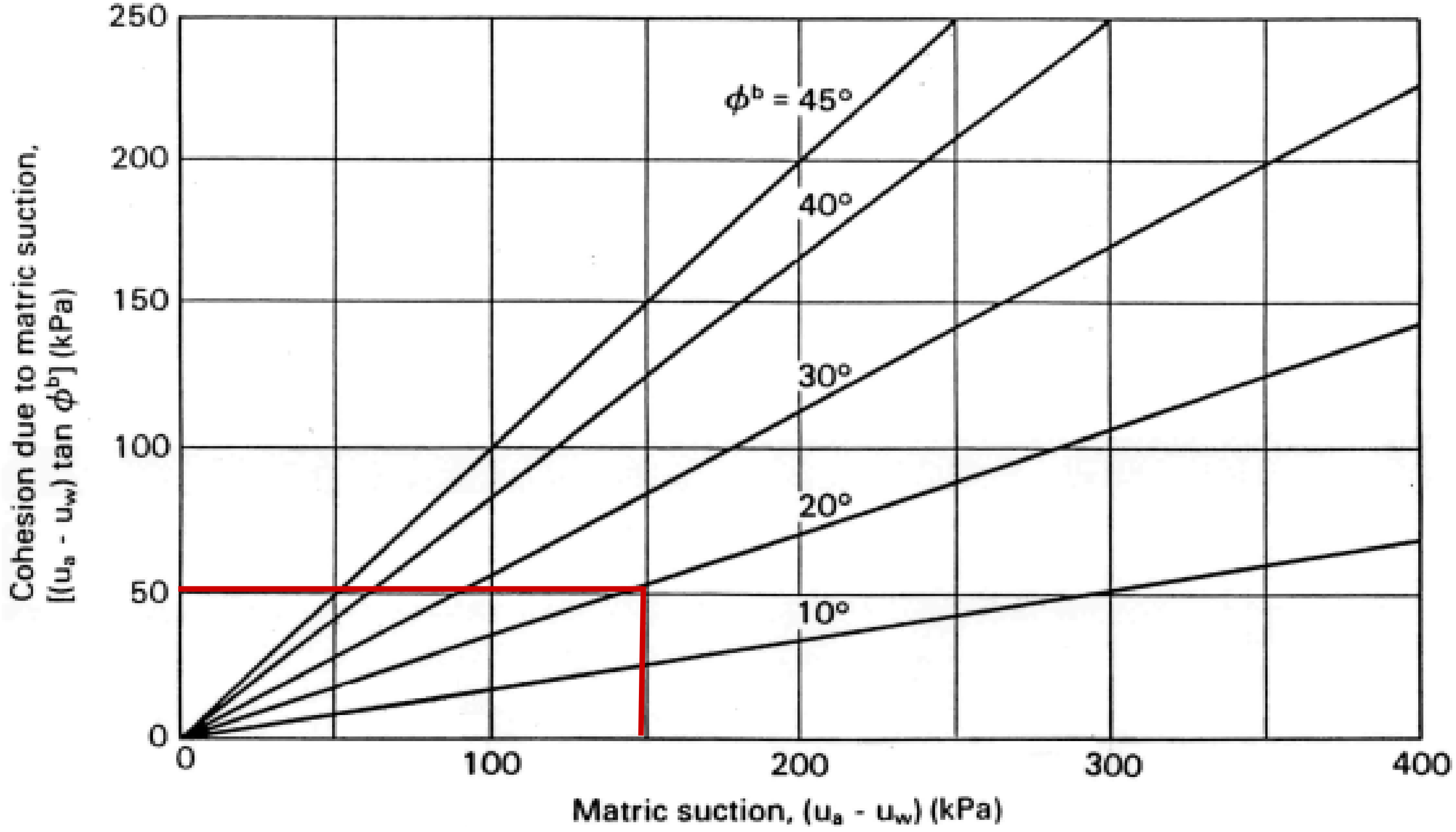
***A grid of pressures has become the most satisfactory way to input pore-water pressures with the utilization of new computer technologies***



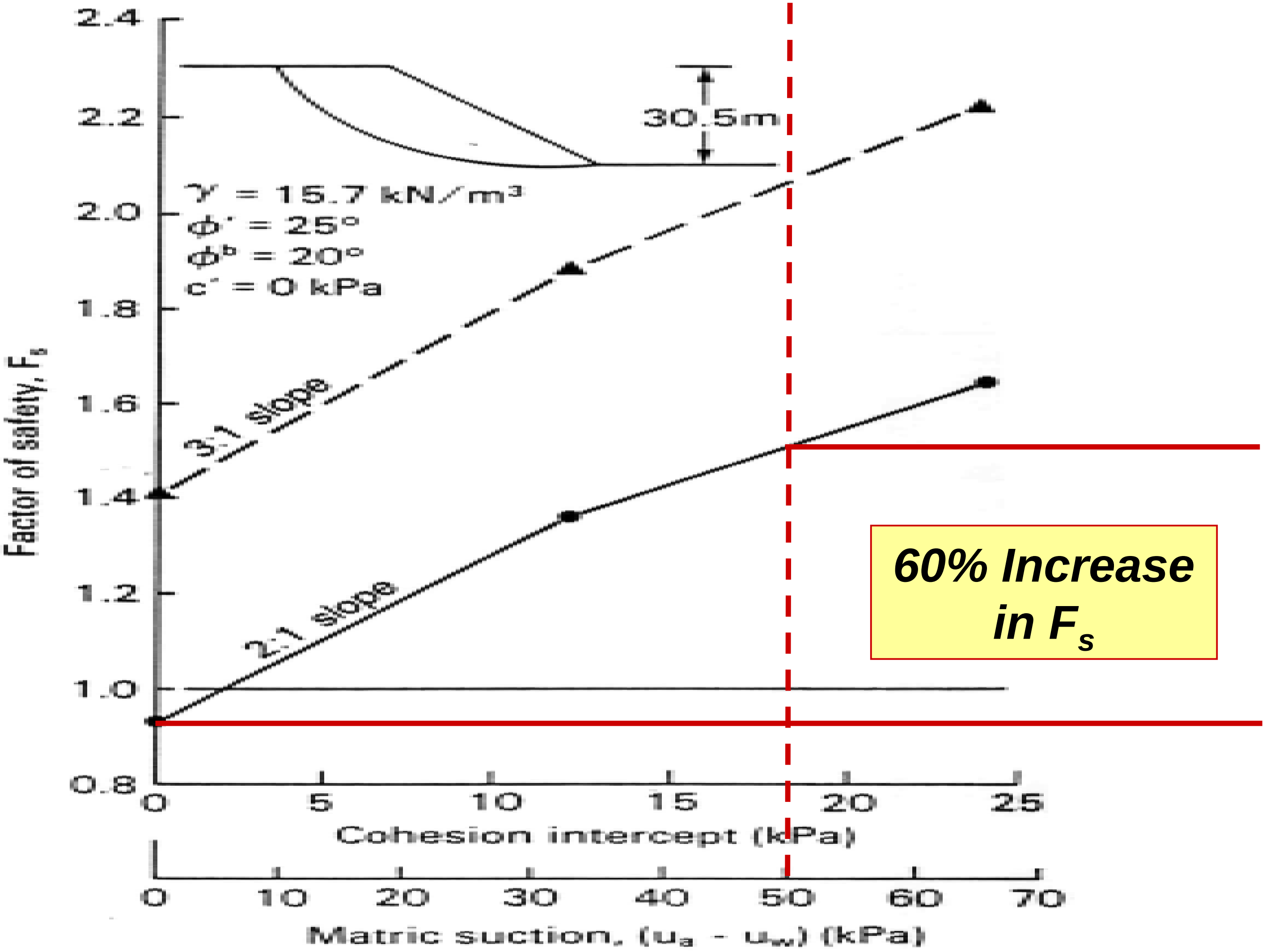
***Utilize the interpolation functions from the Finite element method to obtain the pore-water pressure***

Grid of pore pressure heads superimposed over the geometry

# *Matric suction can be converted into additional cohesion in the soil*

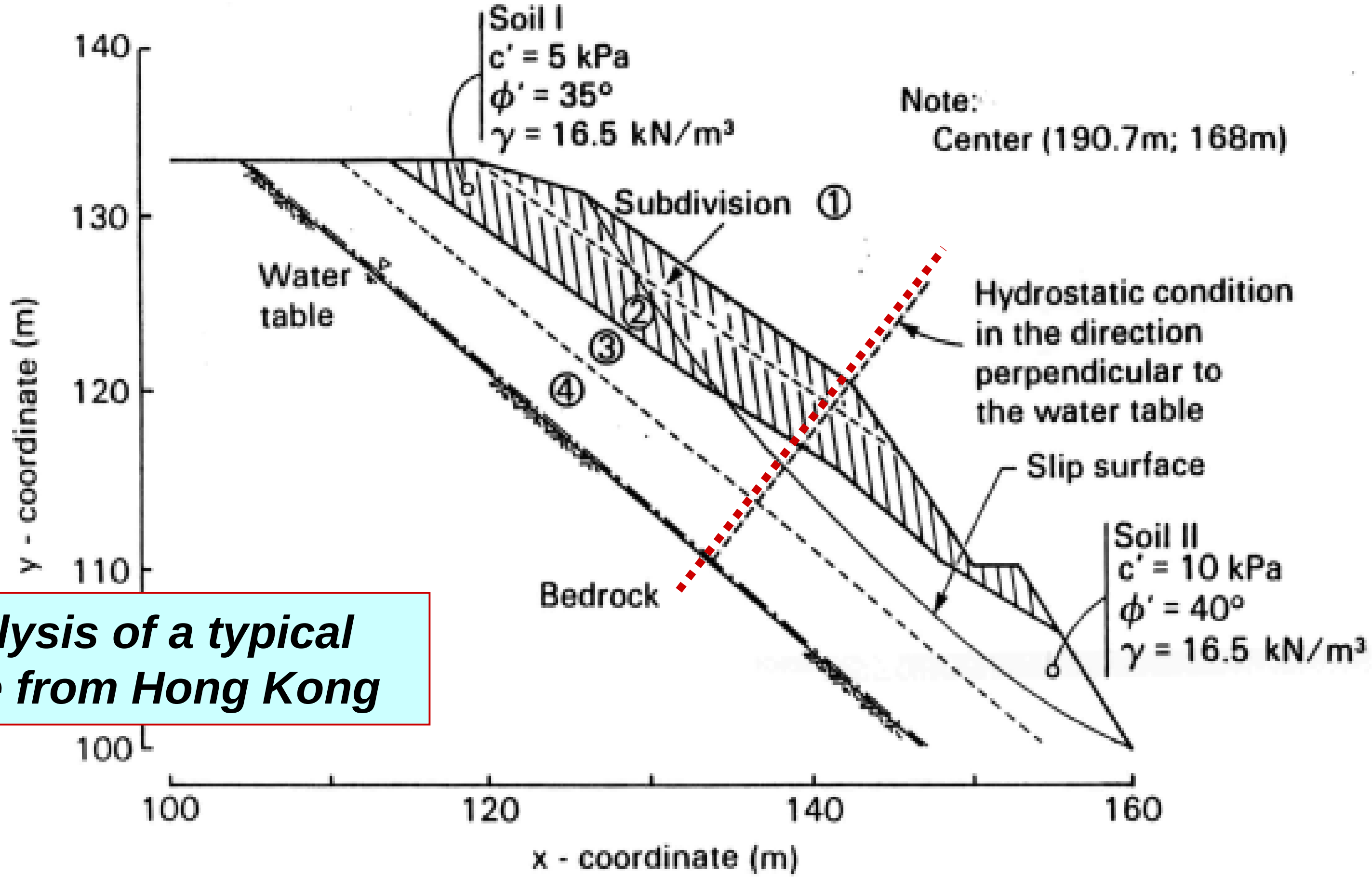


The component of cohesion due to matric suction for various $\phi^b$ angles



**60% Increase in $F_s$**

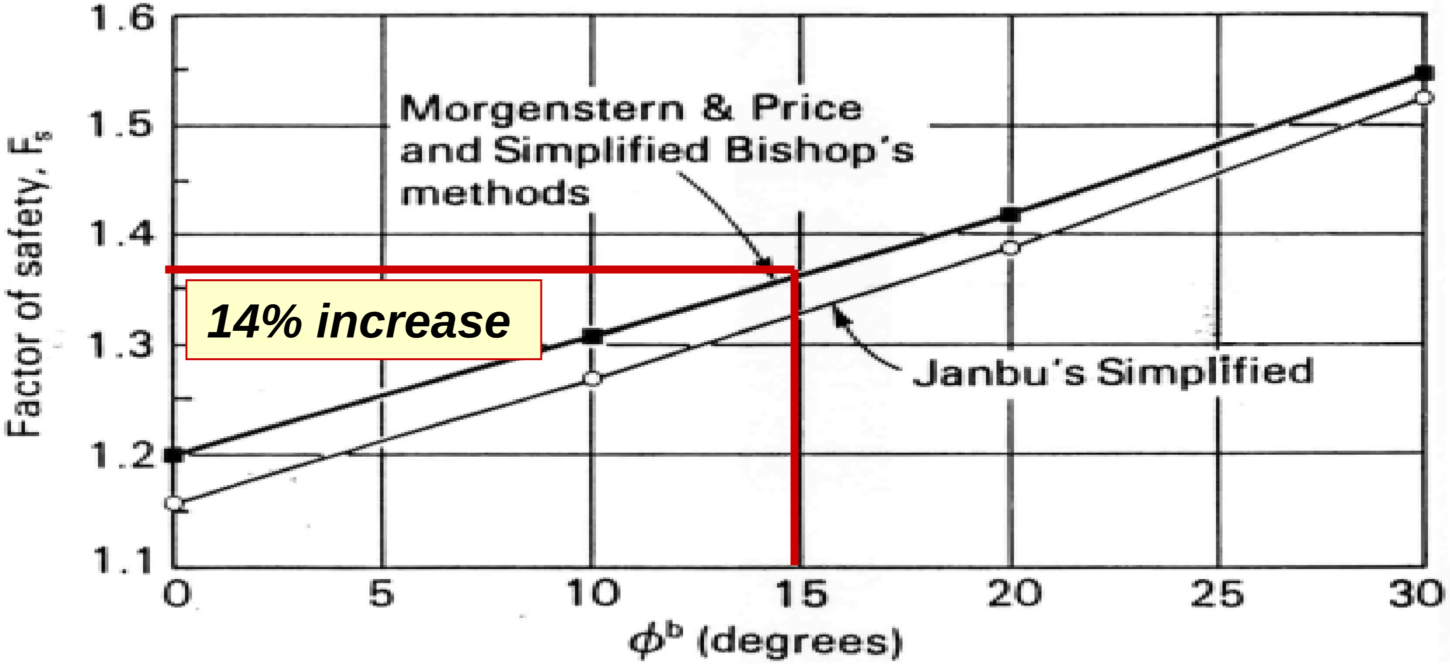
Factor of safety versus matric suction for a simple slope



***Analysis of a typical slope from Hong Kong***

(a) Example of a typical steep slope in Hong Kong (from Sweeney and Robertson, 1979)

Factor of safety of a steep slope versus cohesion increase due to matric suction



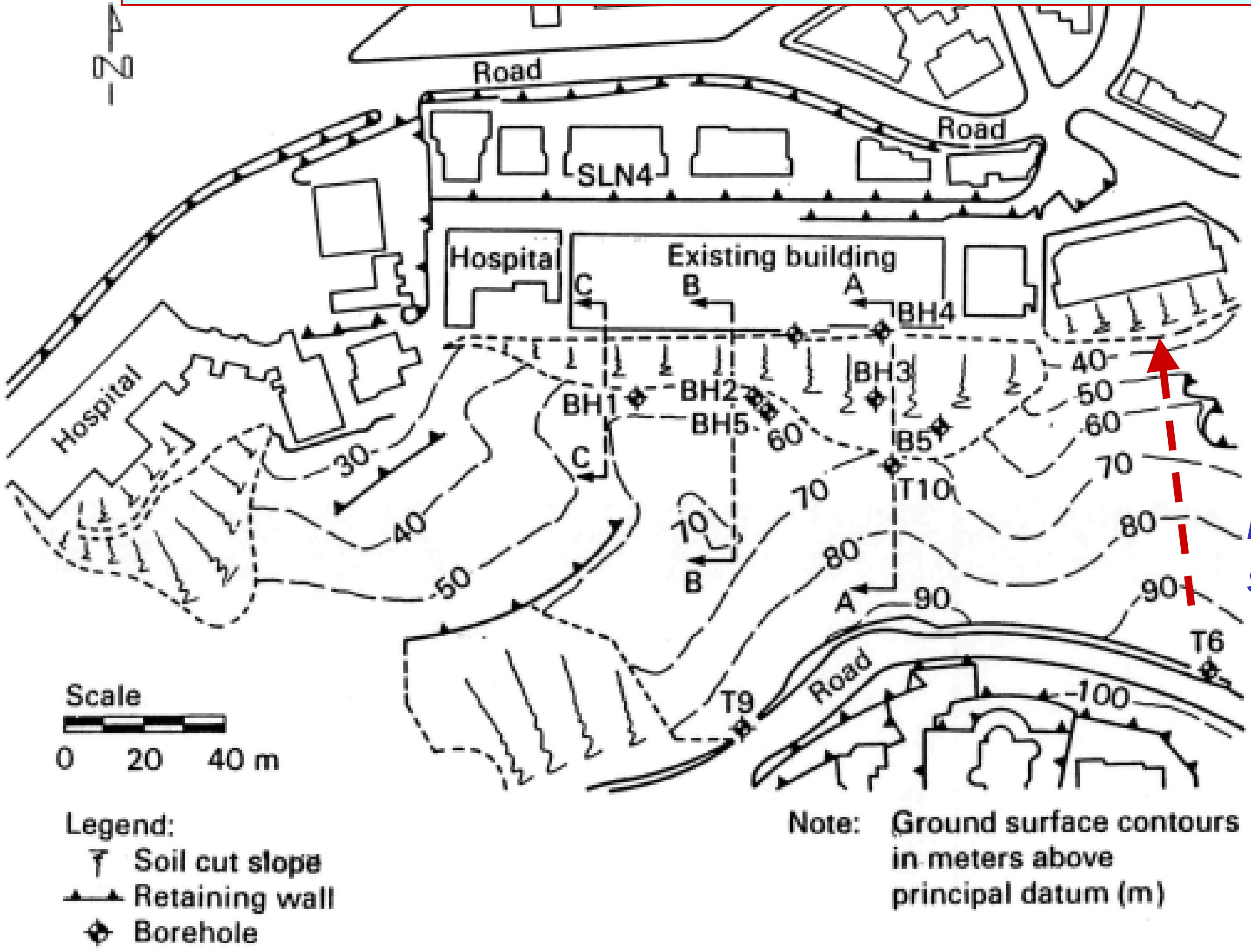
***14% increase***

***All methods of slices show a similar increase in the $F_s$***

(b) Increase in factor of safety due to an increase in $\phi^b$ angle

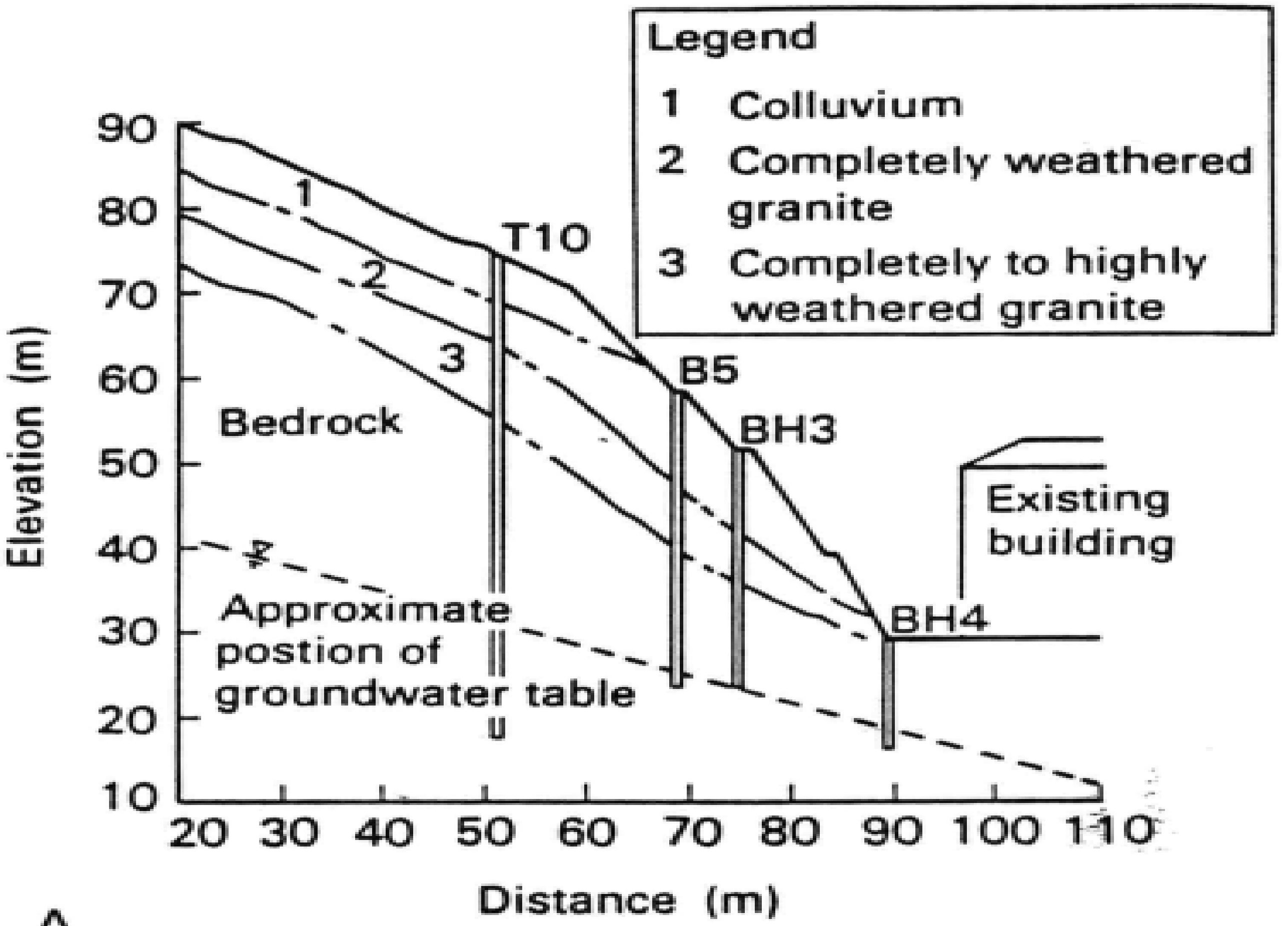
Factor of safety of a steep slope versus cohesion increase due to matric suction (continued)

# Example #1 from Hong Kong showing the analysis for $F_s$ in unsaturated residual soils



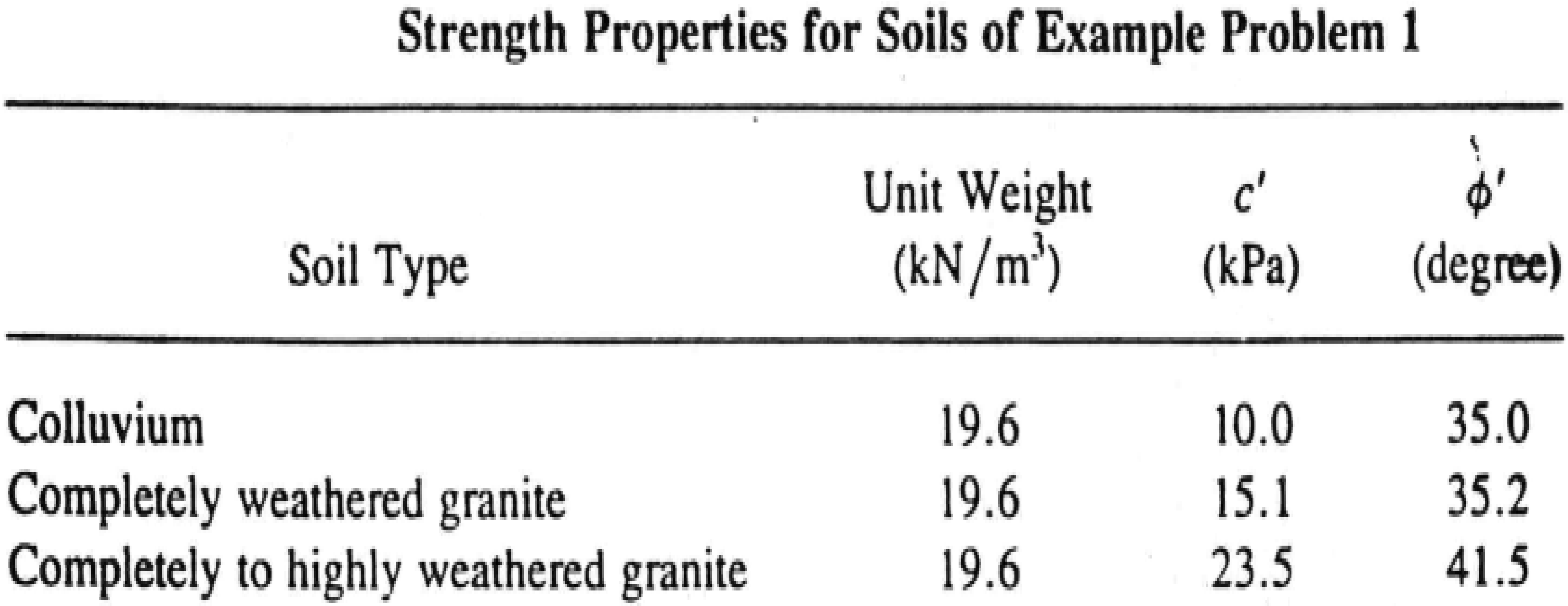
Site plan for example no. 1

**Fung Fai Terrace**



(a) Section A-A

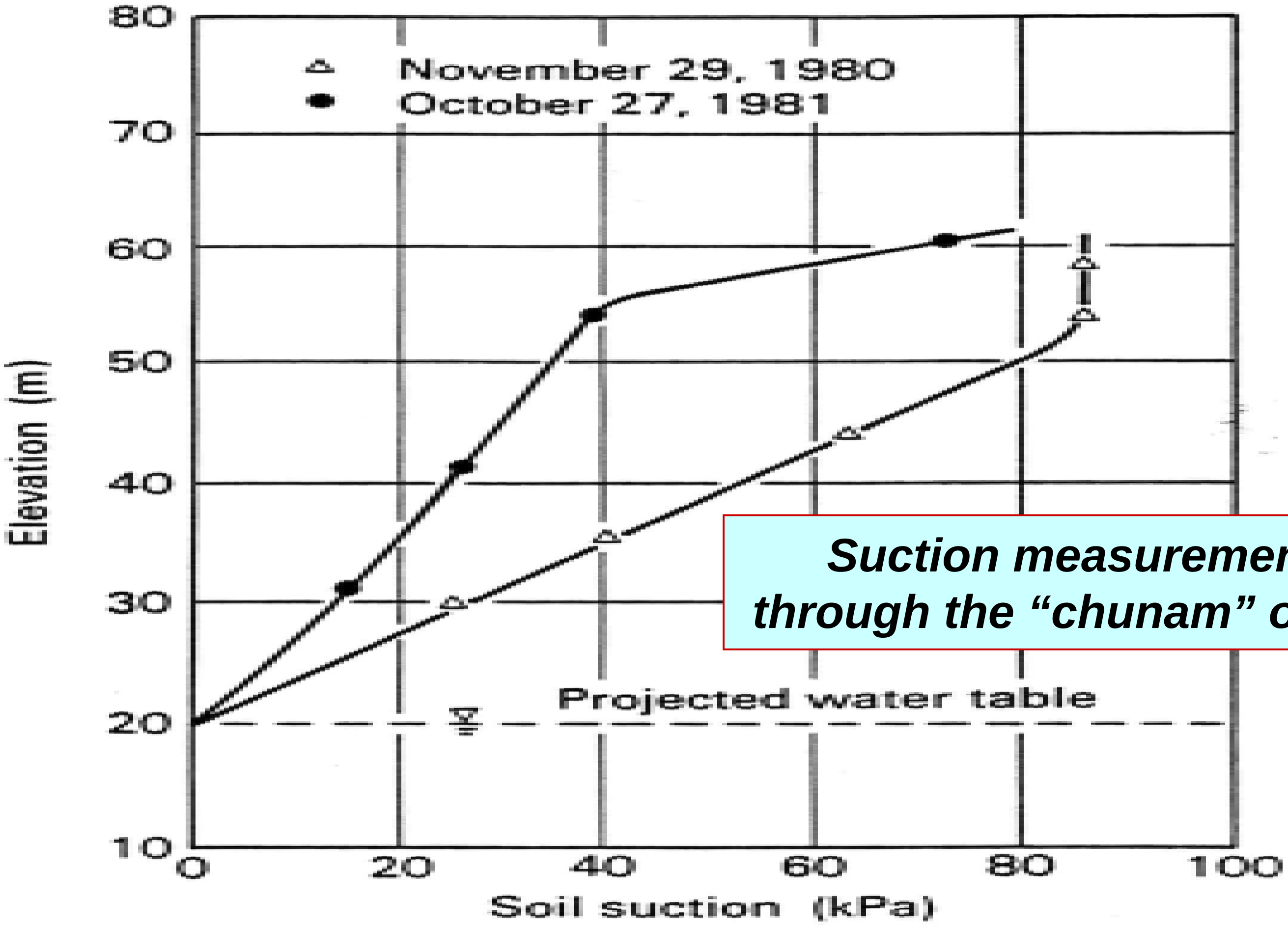
**Fung Fai Terrace**



### Strength Properties for Soils of Example Problem 1

| Soil Type | Unit Weight ($kN/m^3$) | $c'$ (kPa) | $\phi'$ (degree) |
|---|---|---|---|
| Colluvium | 19.6 | 10.0 | 35.0 |
| Completely weathered granite | 19.6 | 15.1 | 35.2 |
| Completely to highly weathered granite | 19.6 | 23.5 | 41.5 |

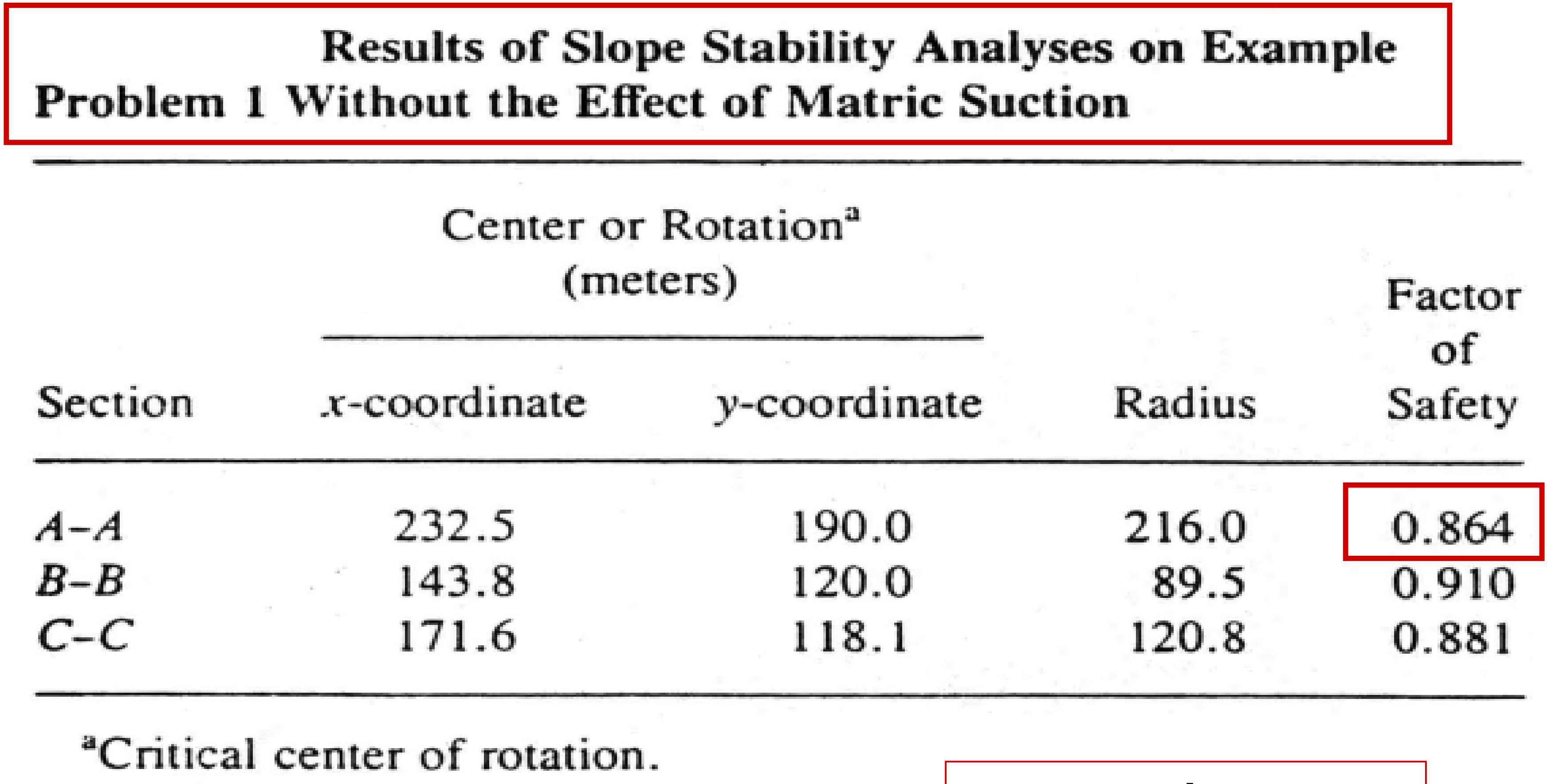
***Extensive shear strength testing shows high angles of internal friction***



***Suction measurements made through the "chunam" on two dates***

***Fung Fai Terrace***

Insitu measurements of matric suction near section A-A for example no. 1 (from Sweeney, 1982)

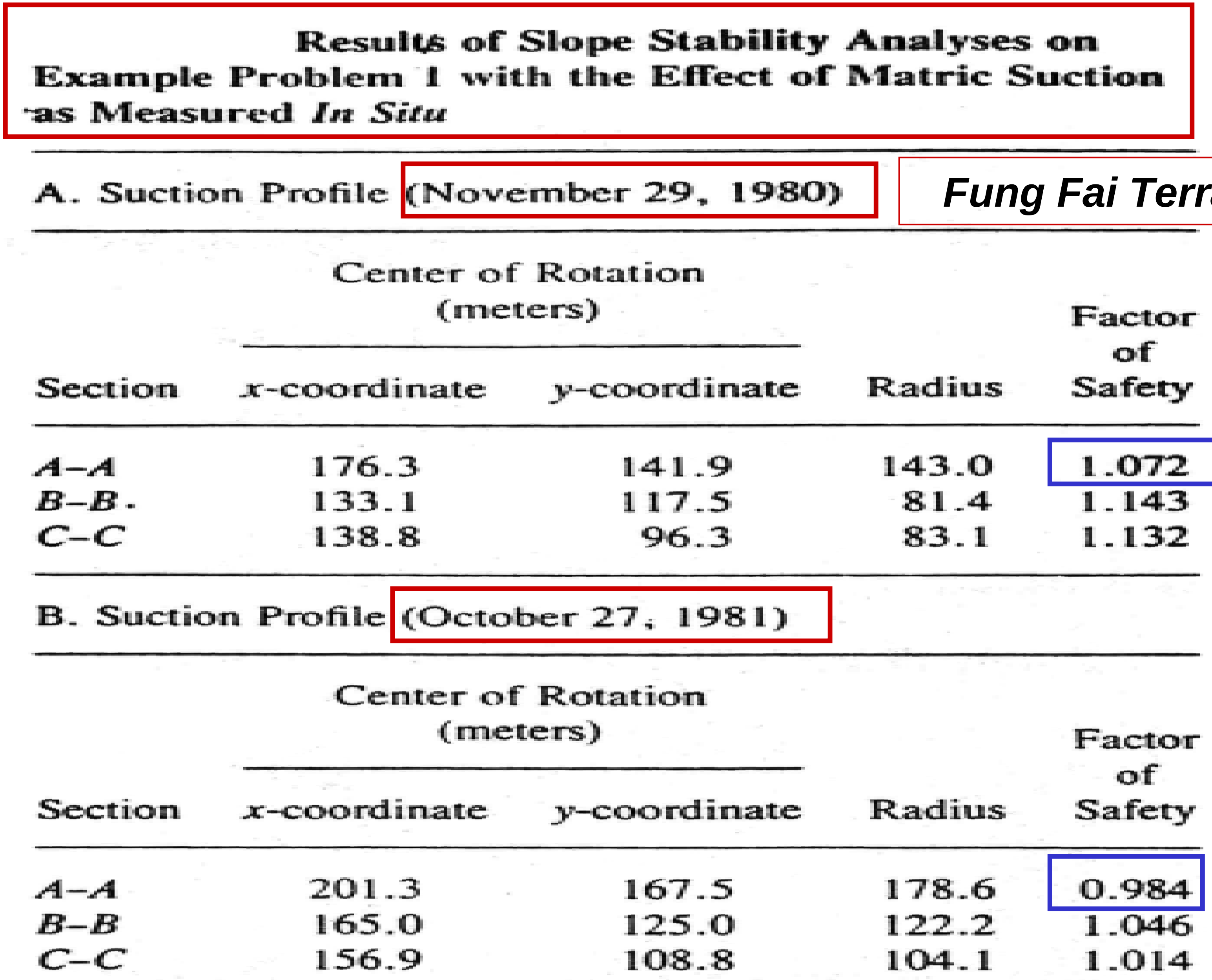


**Results of Slope Stability Analyses on Example Problem 1 Without the Effect of Matric Suction**

| Section | Center or Rotation[a] (meters) |  | Radius | Factor of Safety |
|---|---|---|---|---|
|  | x-coordinate | y-coordinate |  |  |
| A–A | 232.5 | 190.0 | 216.0 | 0.864 |
| B–B | 143.8 | 120.0 | 89.5 | 0.910 |
| C–C | 171.6 | 118.1 | 120.8 | 0.881 |

[a] Critical center of rotation.

*Fung Fai Terrace*



**Results of Slope Stability Analyses on Example Problem 1 with the Effect of Matric Suction as Measured *In Situ***

A. Suction Profile (November 29, 1980)

Fung Fai Terrace

| Section | Center of Rotation (meters) x-coordinate | Center of Rotation (meters) y-coordinate | Radius | Factor of Safety |
|---|---|---|---|---|
| *A–A* | 176.3 | 141.9 | 143.0 | 1.072 |
| *B–B* | 133.1 | 117.5 | 81.4 | 1.143 |
| C–C | 138.8 | 96.3 | 83.1 | 1.132 |

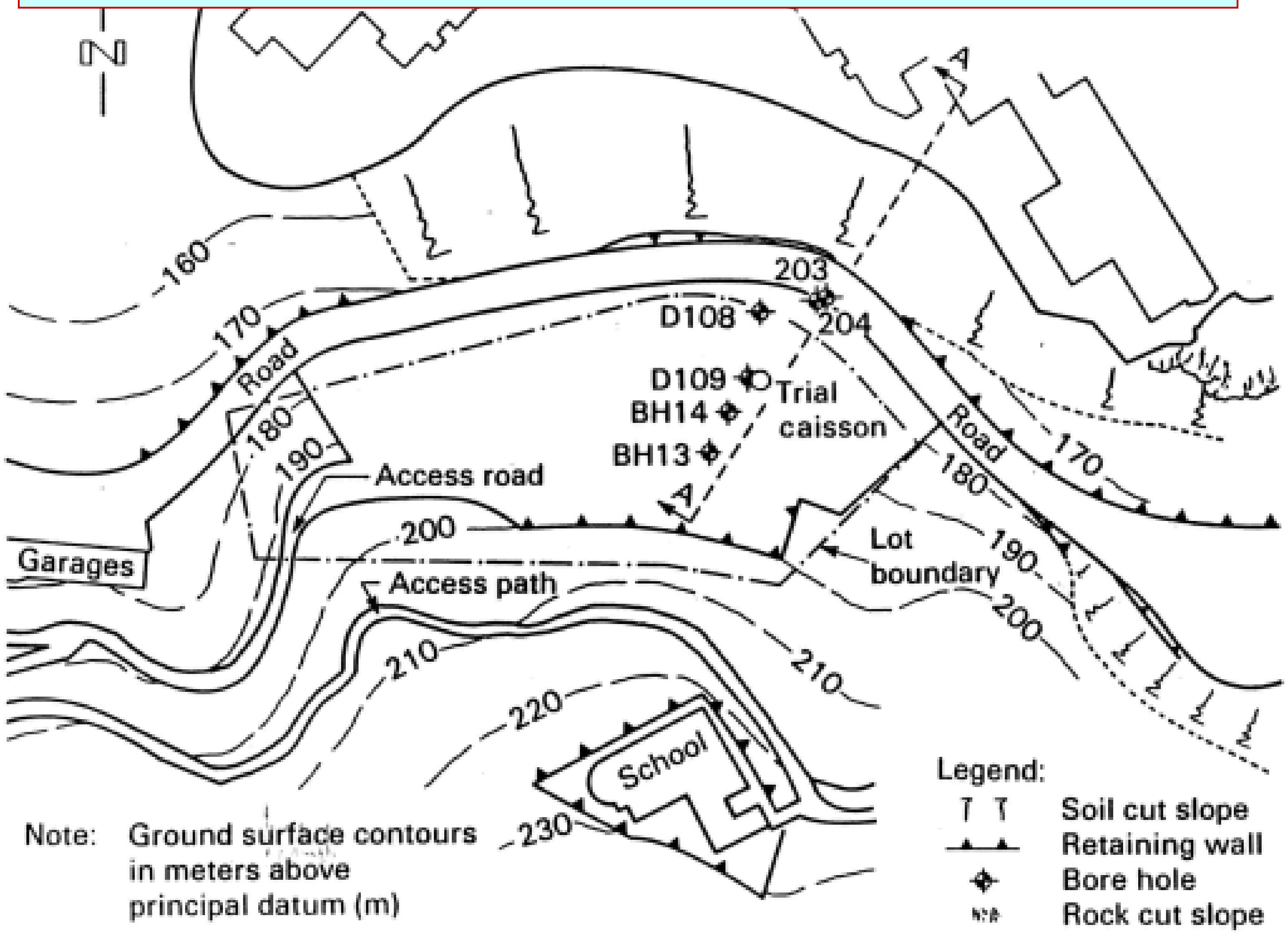
24%

B. Suction Profile (October 27, 1981)

| Section | Center of Rotation (meters) x-coordinate | Center of Rotation (meters) y-coordinate | Radius | Factor of Safety |
|---|---|---|---|---|
| *A–A* | 201.3 | 167.5 | 178.6 | 0.984 |
| *B–B* | 165.0 | 125.0 | 122.2 | 1.046 |
| C–C | 156.9 | 108.8 | 104.1 | 1.014 |

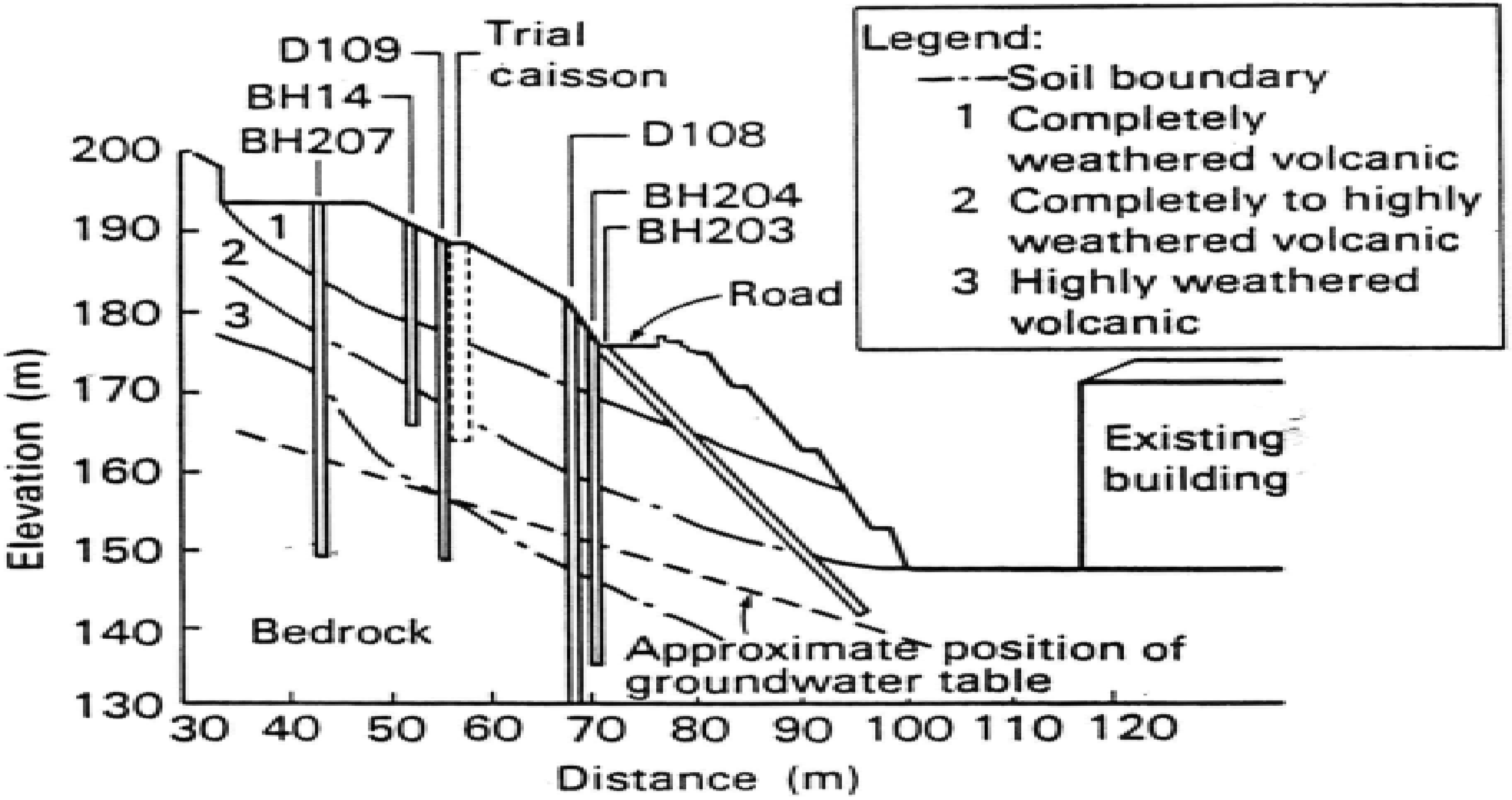
12%

# Example #2 from Hong Kong showing the analysis for $F_s$ in unsaturated residual soils

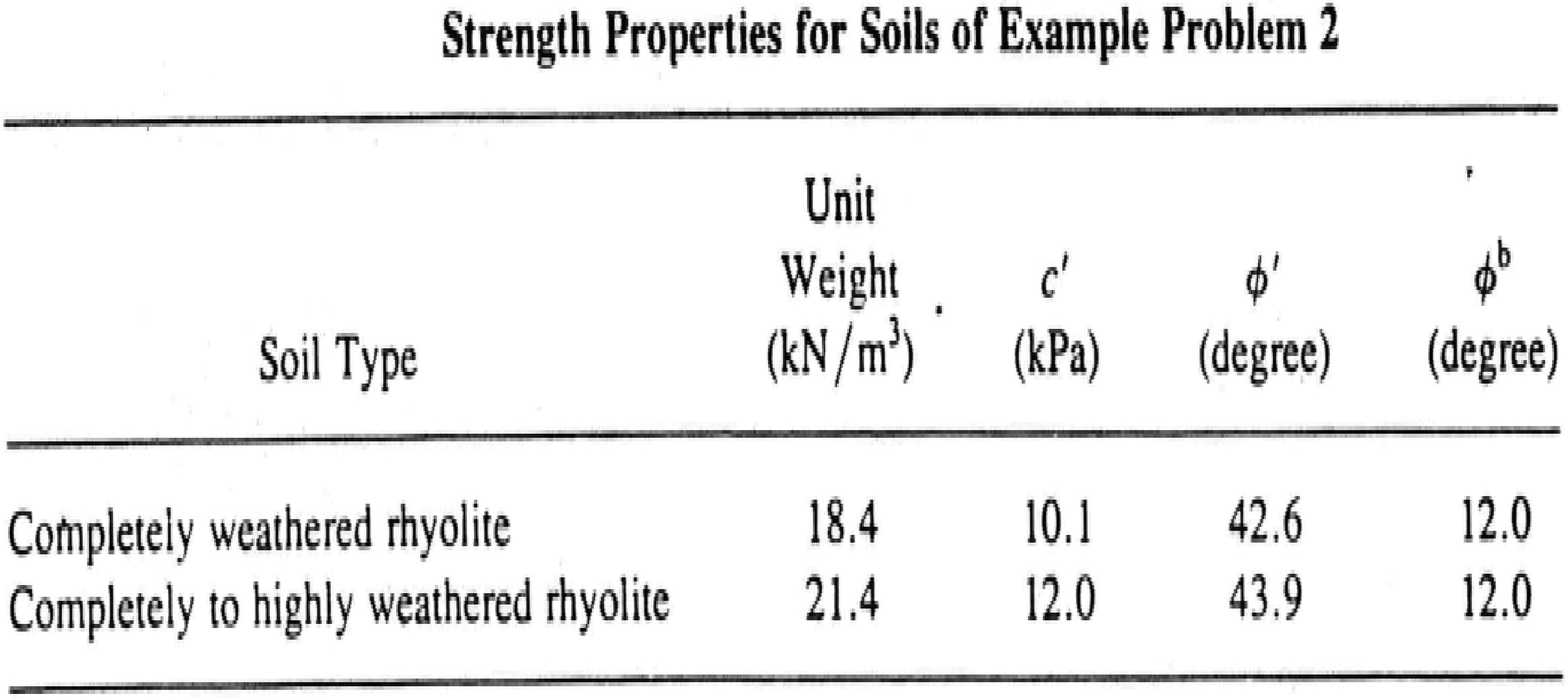


**Thorpe Manor**

Site plan for example no. 2



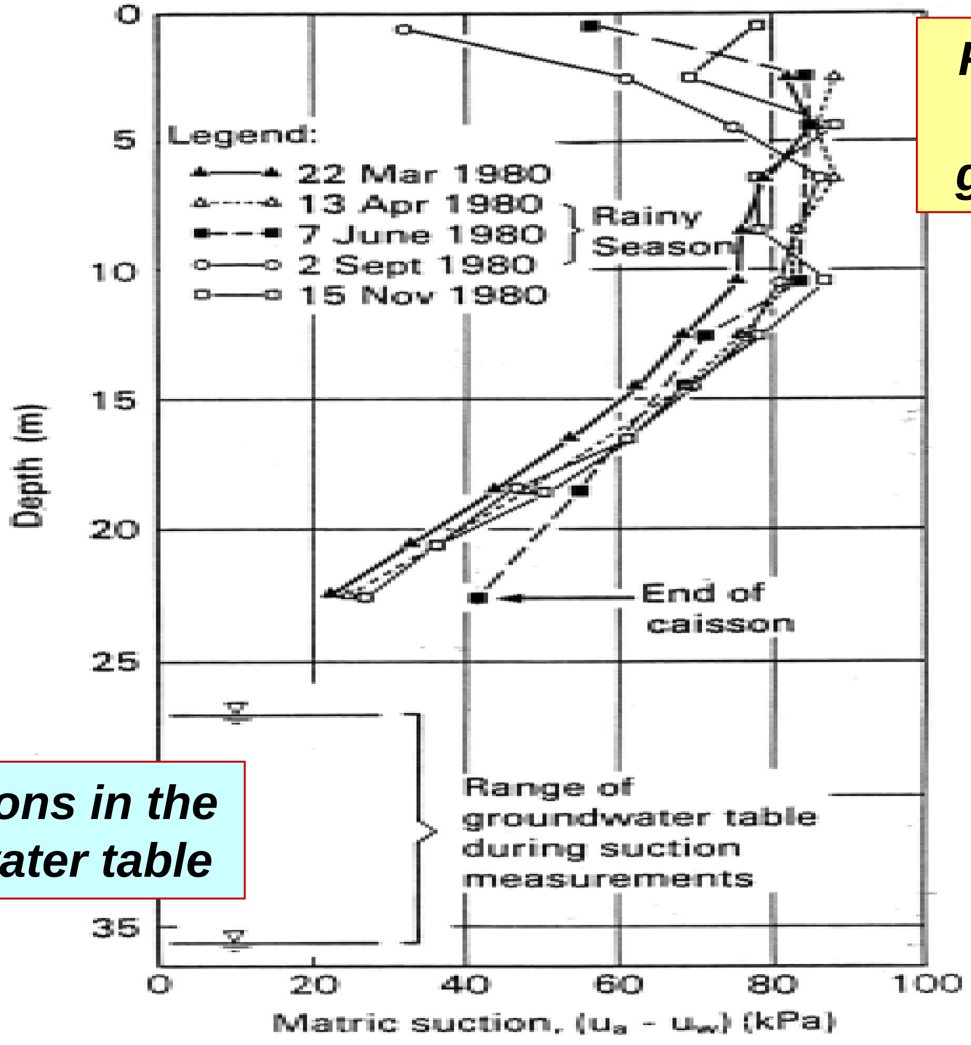
Section A-A for example no. 2



Strength Properties for Soils of Example Problem 2

| Soil Type | Unit Weight (kN/m$^3$) | $c'$ (kPa) | $\phi'$ (degree) | $\phi^b$ (degree) |
|---|---|---|---|---|
| Completely weathered rhyolite | 18.4 | 10.1 | 42.6 | 12.0 |
| Completely to highly weathered rhyolite | 21.4 | 12.0 | 43.9 | 12.0 |

***Note:* high angles of internal friction for the decomposed rhyolite**



***Fluctuations of suction near ground surface***

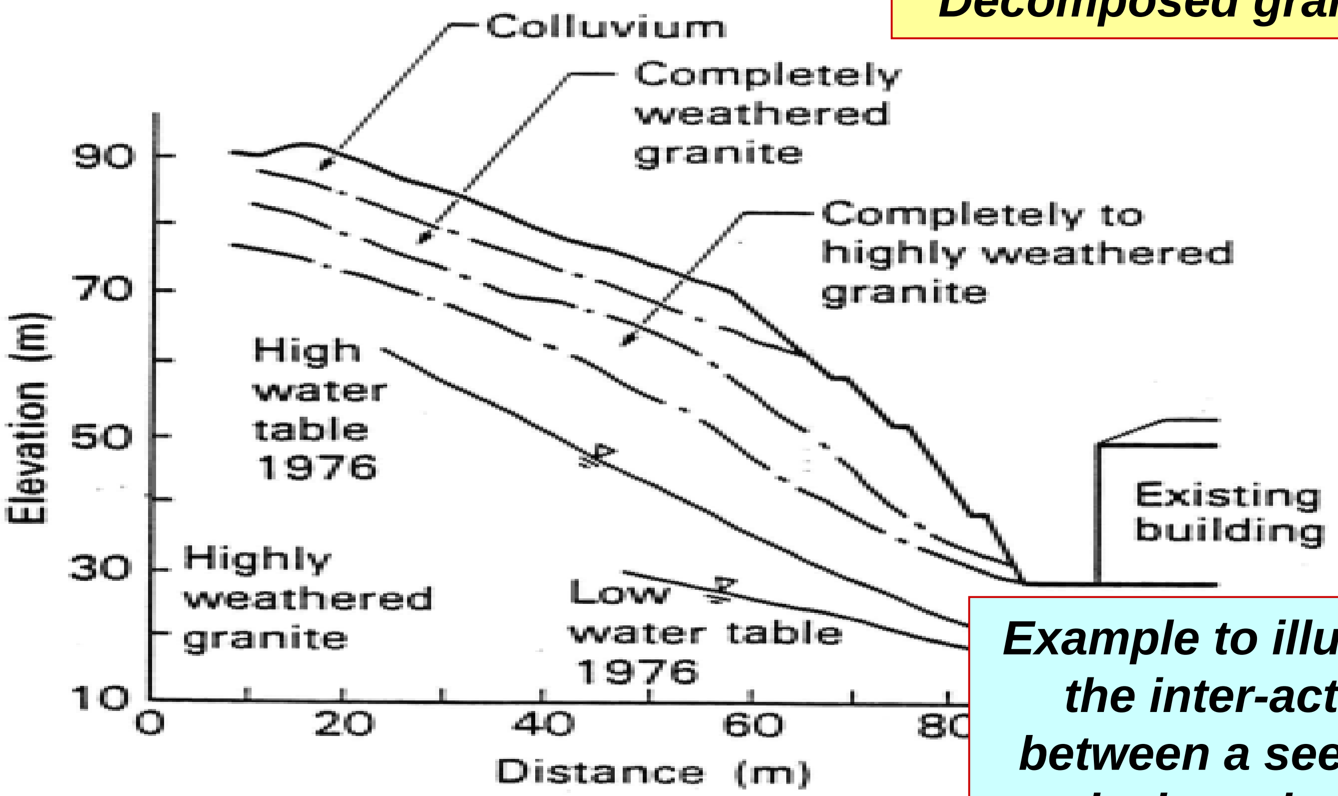
***Fluctuations in the groundwater table***

Insitu measurements of matric suction throughout 1980 for example no. 2 (from Sweeney, 1982)

# The "extended shear strength" method

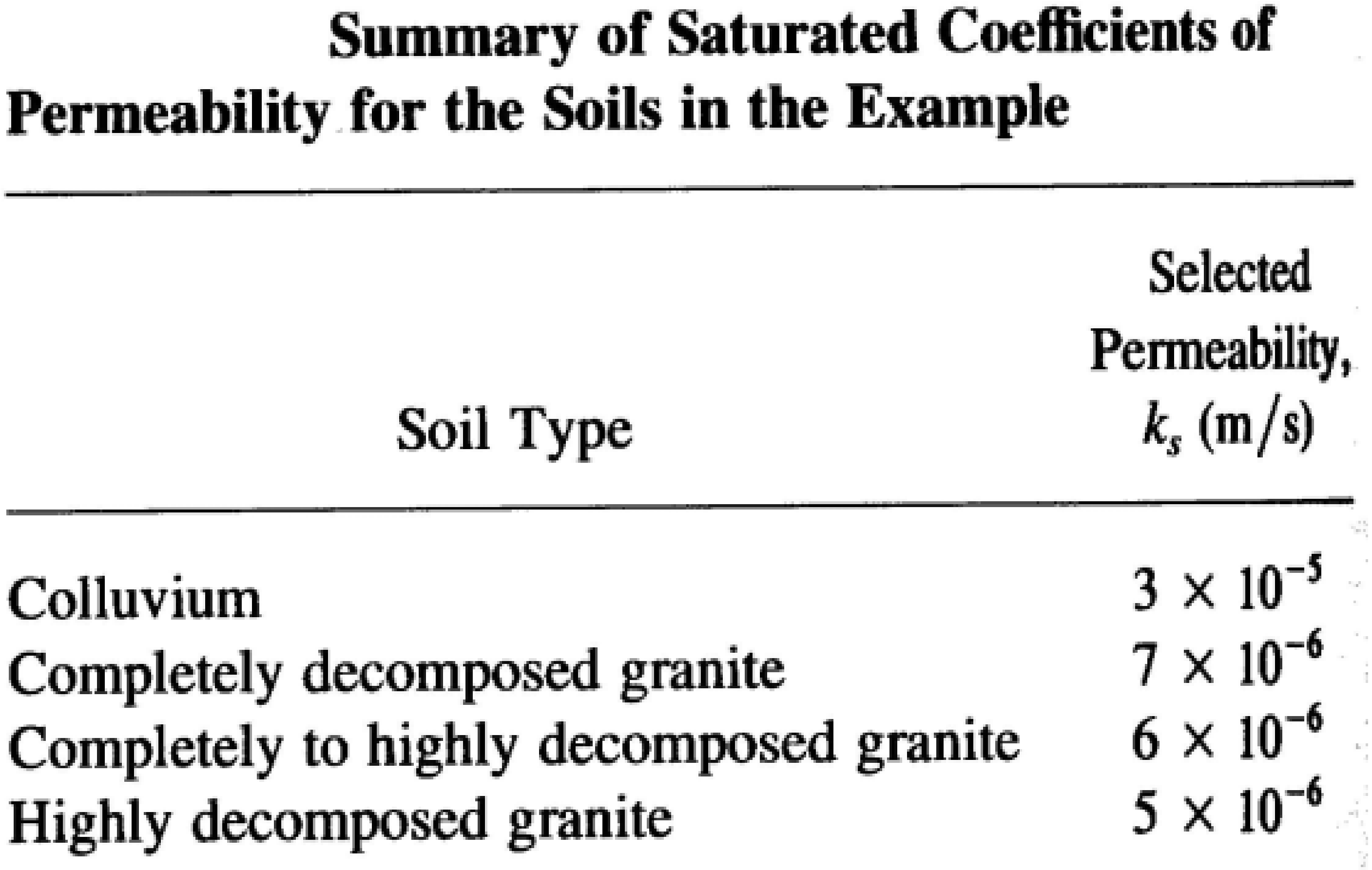
## General layout of problems and soil properties

***Decomposed granite***



***Example to illustrate the inter-action between a seepage analysis and a slope stability analysis***

Cross-section of a steep slope of residual soil

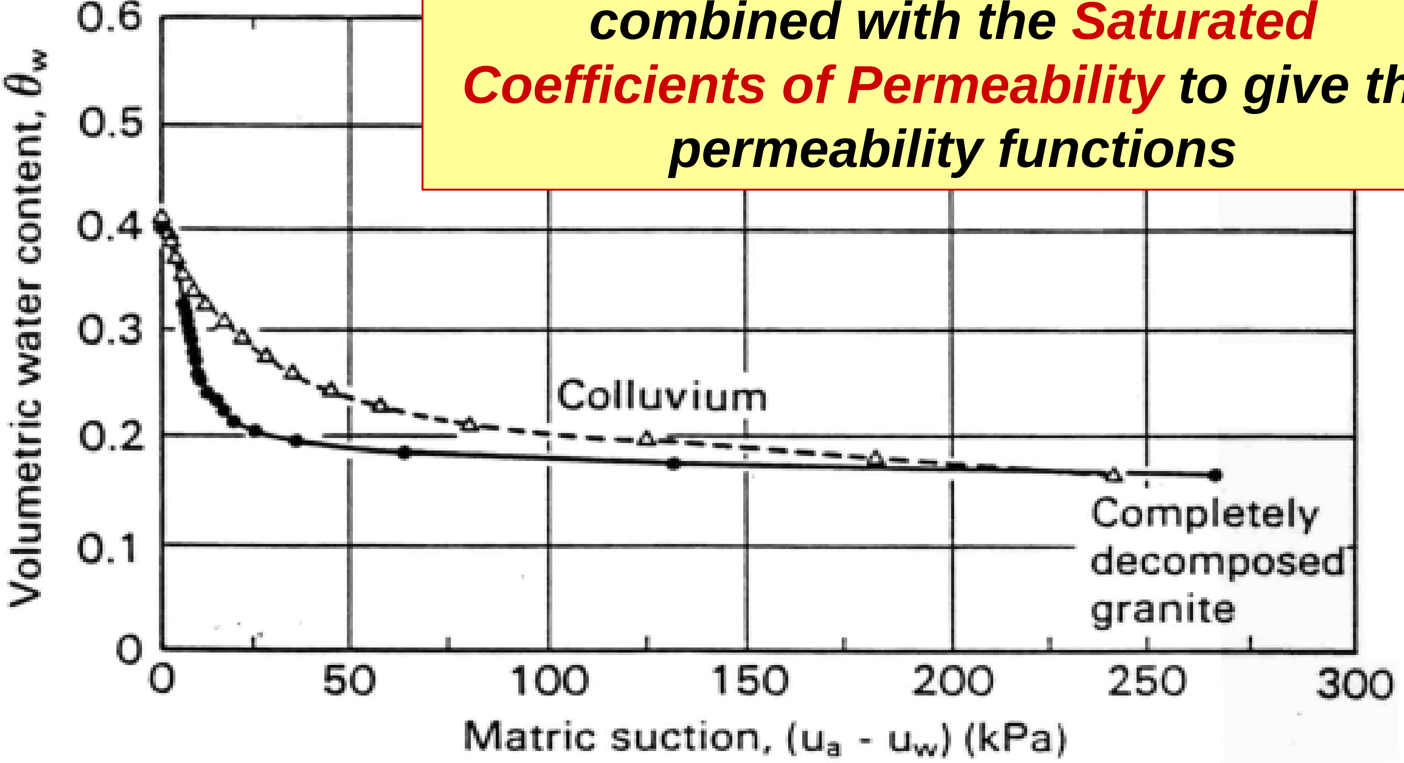


## Summary of Saturated Coefficients of Permeability for the Soils in the Example

| Soil Type | Selected Permeability, $k_s$ (m/s) |
|---|---|
| Colluvium | $3 \times 10^{-5}$ |
| Completely decomposed granite | $7 \times 10^{-6}$ |
| Completely to highly decomposed granite | $6 \times 10^{-6}$ |
| Highly decomposed granite | $5 \times 10^{-6}$ |

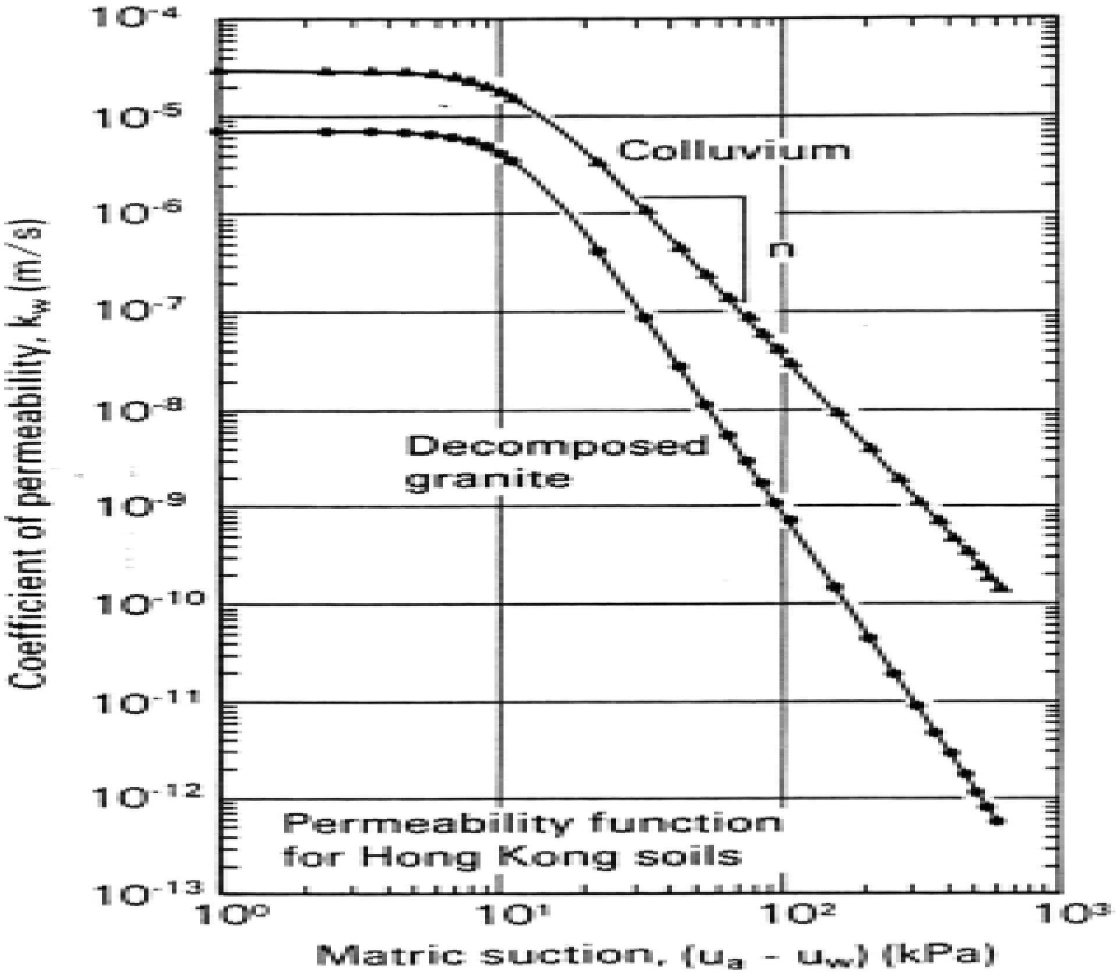
***Measured saturated coefficients of permeabilities***

***Soil-Water Characteristic Curves* can be combined with the *Saturated Coefficients of Permeability* to give the permeability functions**



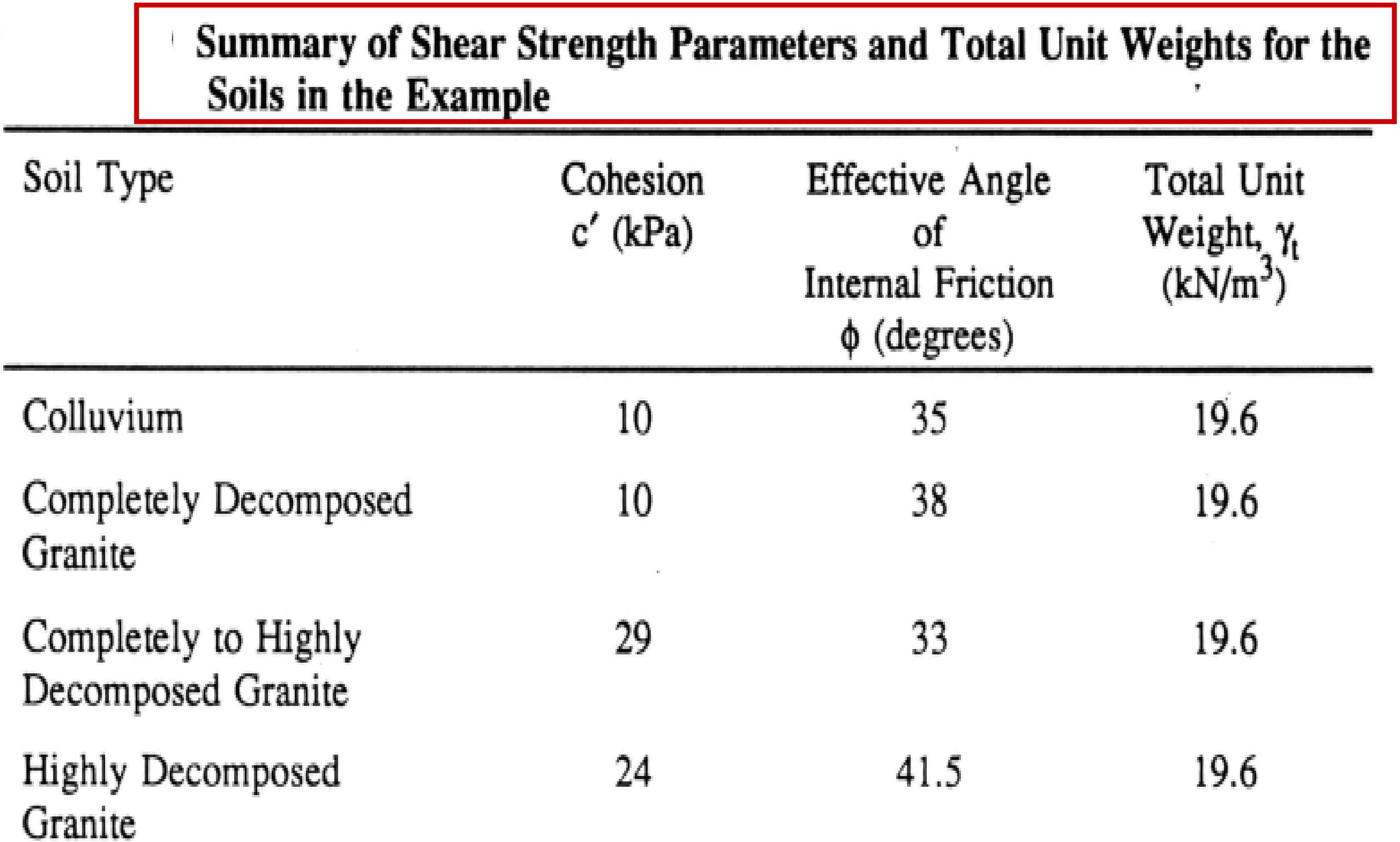
Soil-water characteristic curves for the completely decomposed granite and the colluvium

# *Permeability functions for Decomposed Granite and Colluvium*



Unsaturated coefficient of permeability functions for decomposed granite and colluvium

$$k_w = \frac{k_s}{1 + a\left\{\dfrac{(u_a - u_w)}{\rho_w g}\right\}^n}$$

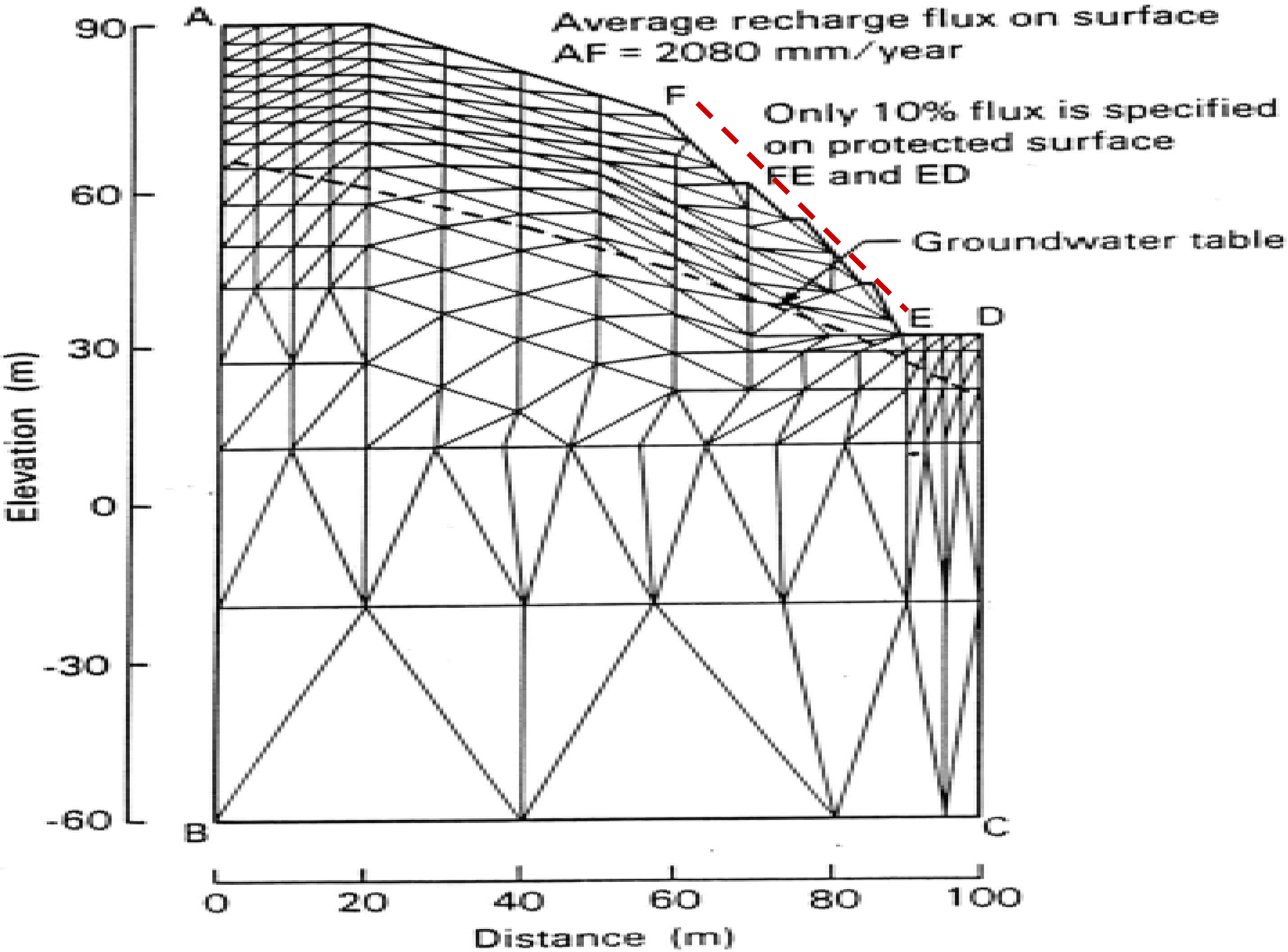


## Summary of Shear Strength Parameters and Total Unit Weights for the Soils in the Example

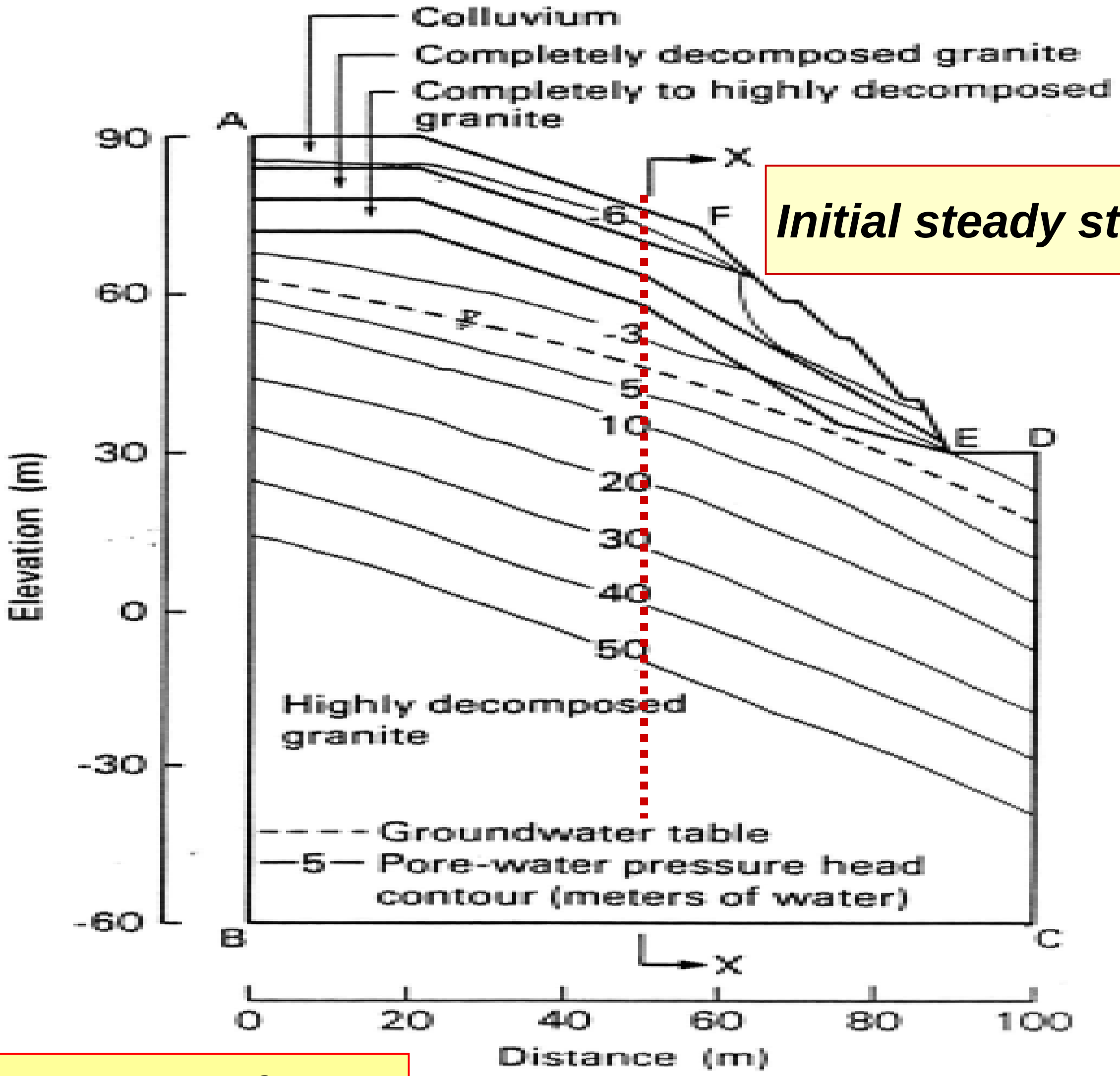
| Soil Type | Cohesion $c'$ (kPa) | Effective Angle of Internal Friction $\phi$ (degrees) | Total Unit Weight, $\gamma_t$ (kN/m$^3$) |
|---|---|---|---|
| Colluvium | 10 | 35 | 19.6 |
| Completely Decomposed Granite | 10 | 38 | 19.6 |
| Completely to Highly Decomposed Granite | 29 | 33 | 19.6 |
| Highly Decomposed Granite | 24 | 41.5 | 19.6 |

***Soil parameters used in the parametric study***

# Initial conditions for the seepage analysis

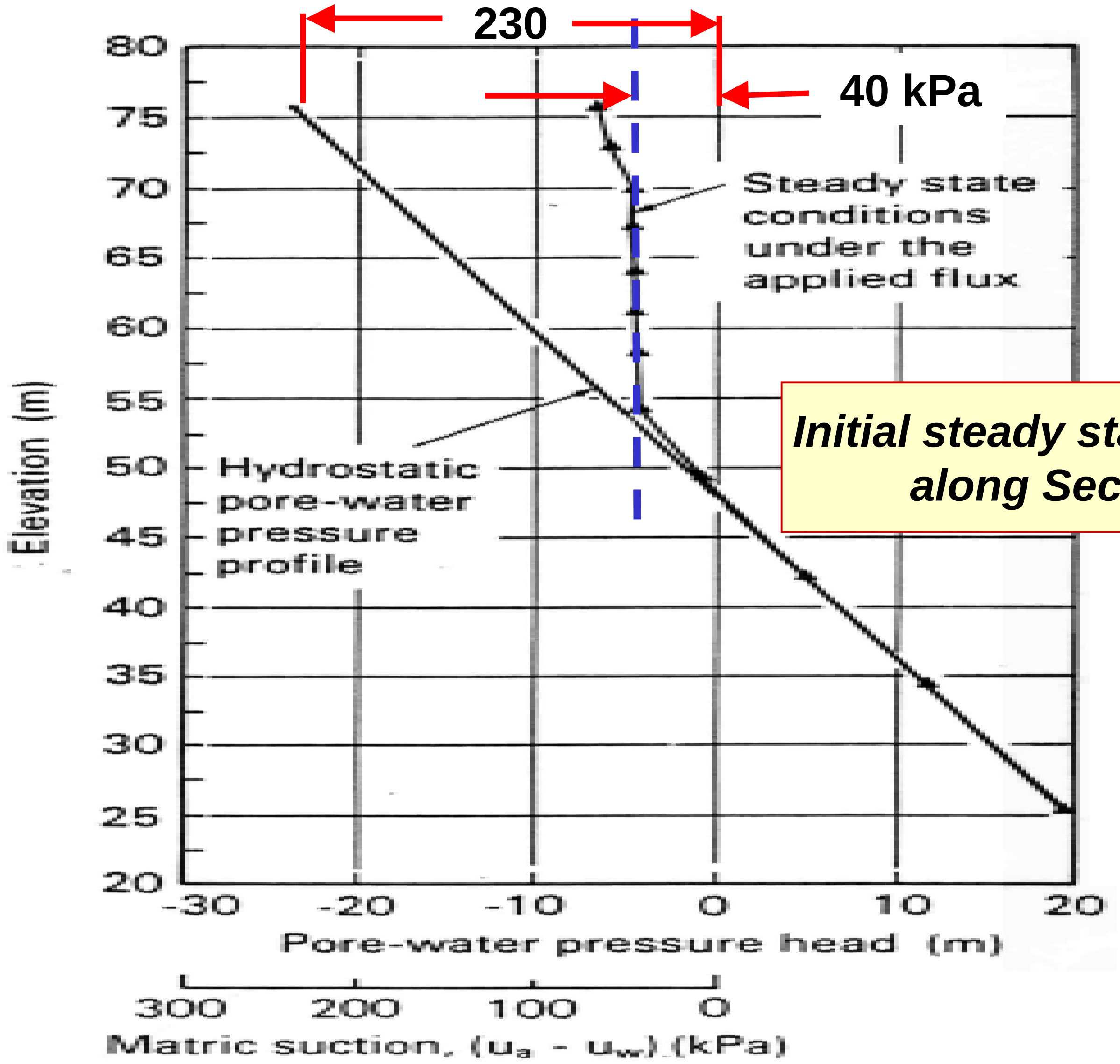


***Initial steady state conditions***



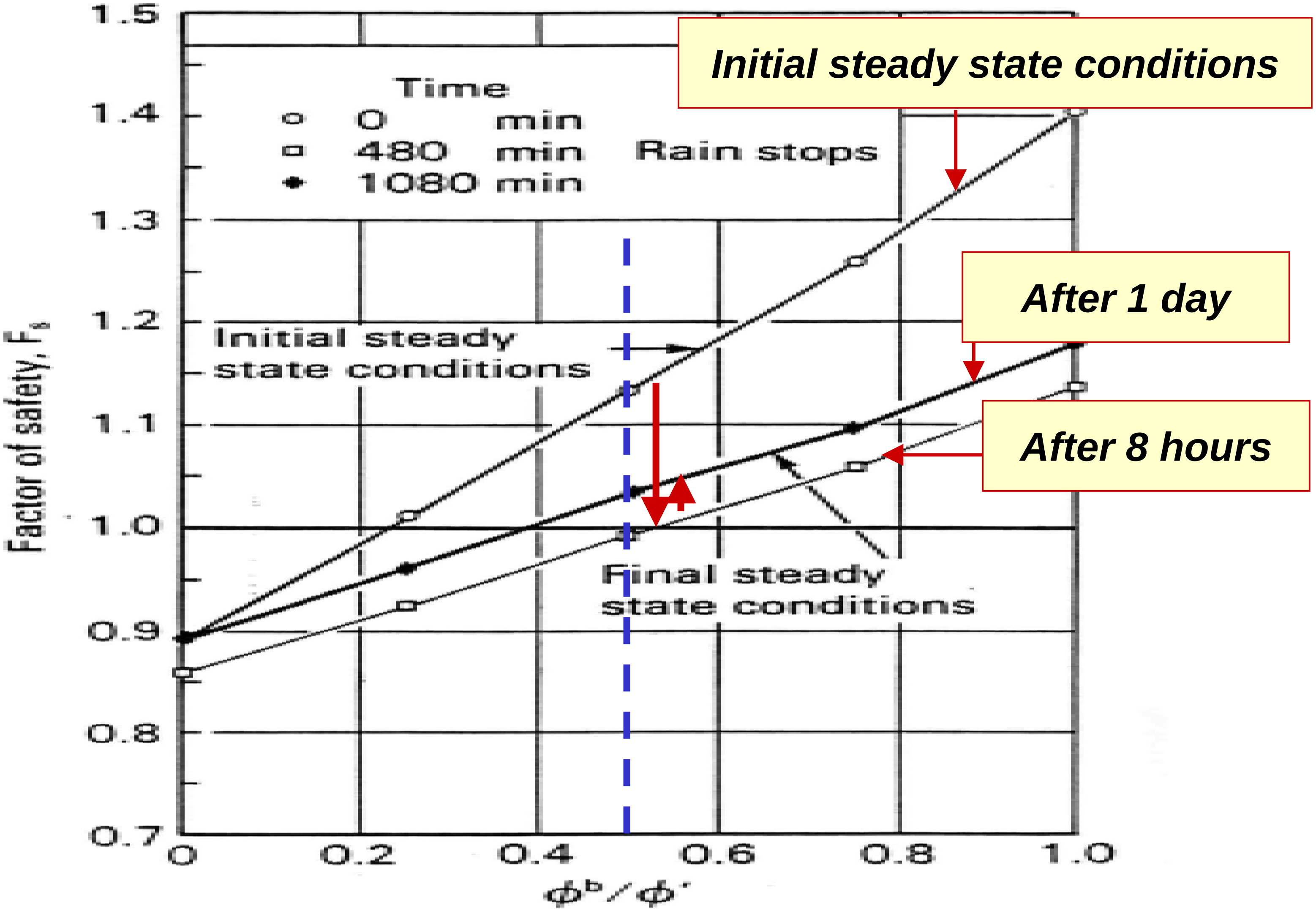
***2080 mm/ year = 6.6 x 10 $^{-8}$ m/s***

Initial groundwater condition and pore-water pressure head contours



***Initial steady state conditions along Section X - X***

Matric suction profiles for section X-X under steady state flux conditions



*Initial steady state conditions*

*After 1 day*

*After 8 hours*

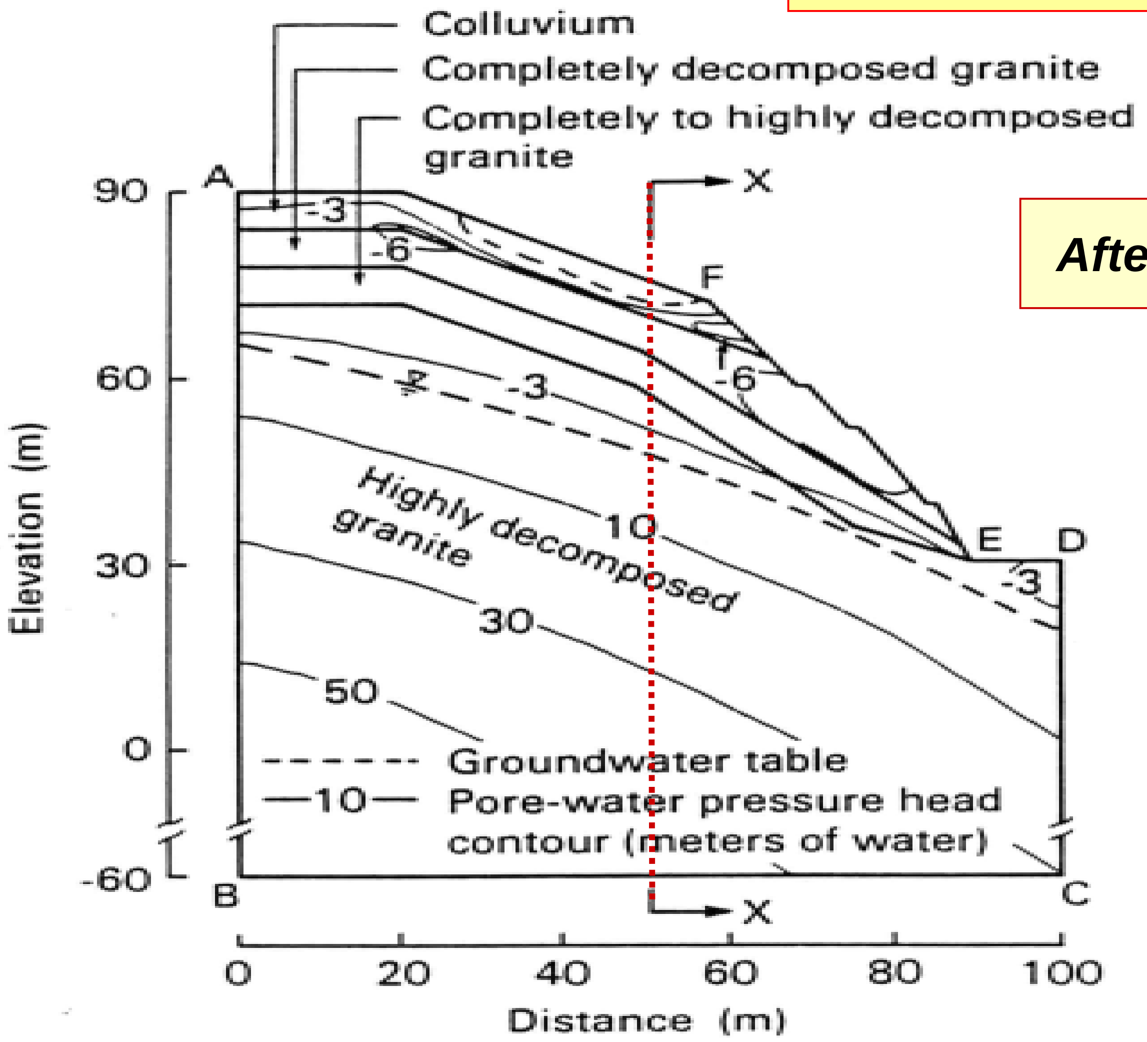
Factors of safety with respect to $\phi^b / \phi'$ for various seepage conditions

**Unsaturated Soil Technology**

# Seepage and slope stability results under high intensity rainfall conditions

***Flux = 1.3 x 10 $^{-5}$ m/s***

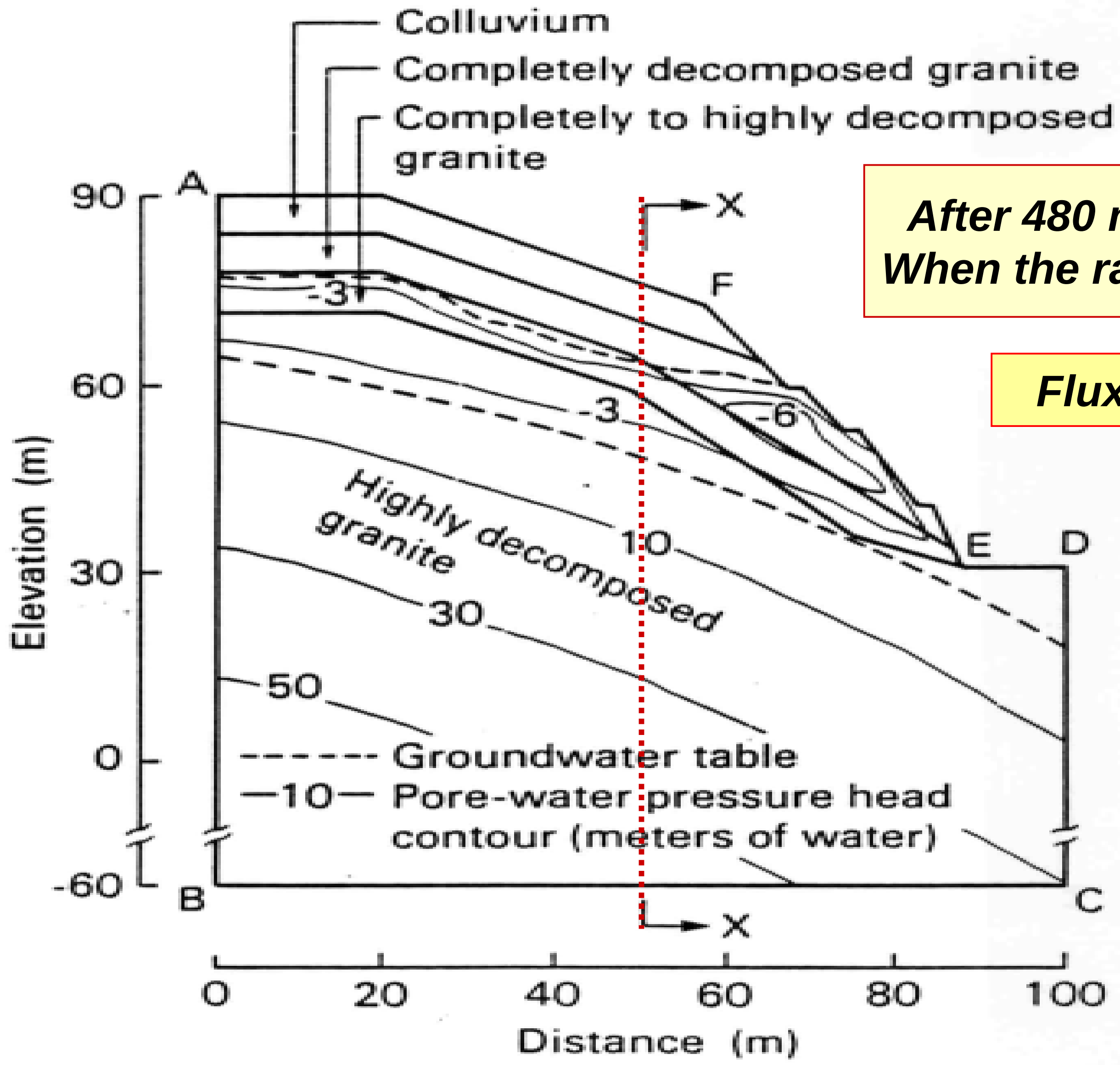
***After 120 minutes***



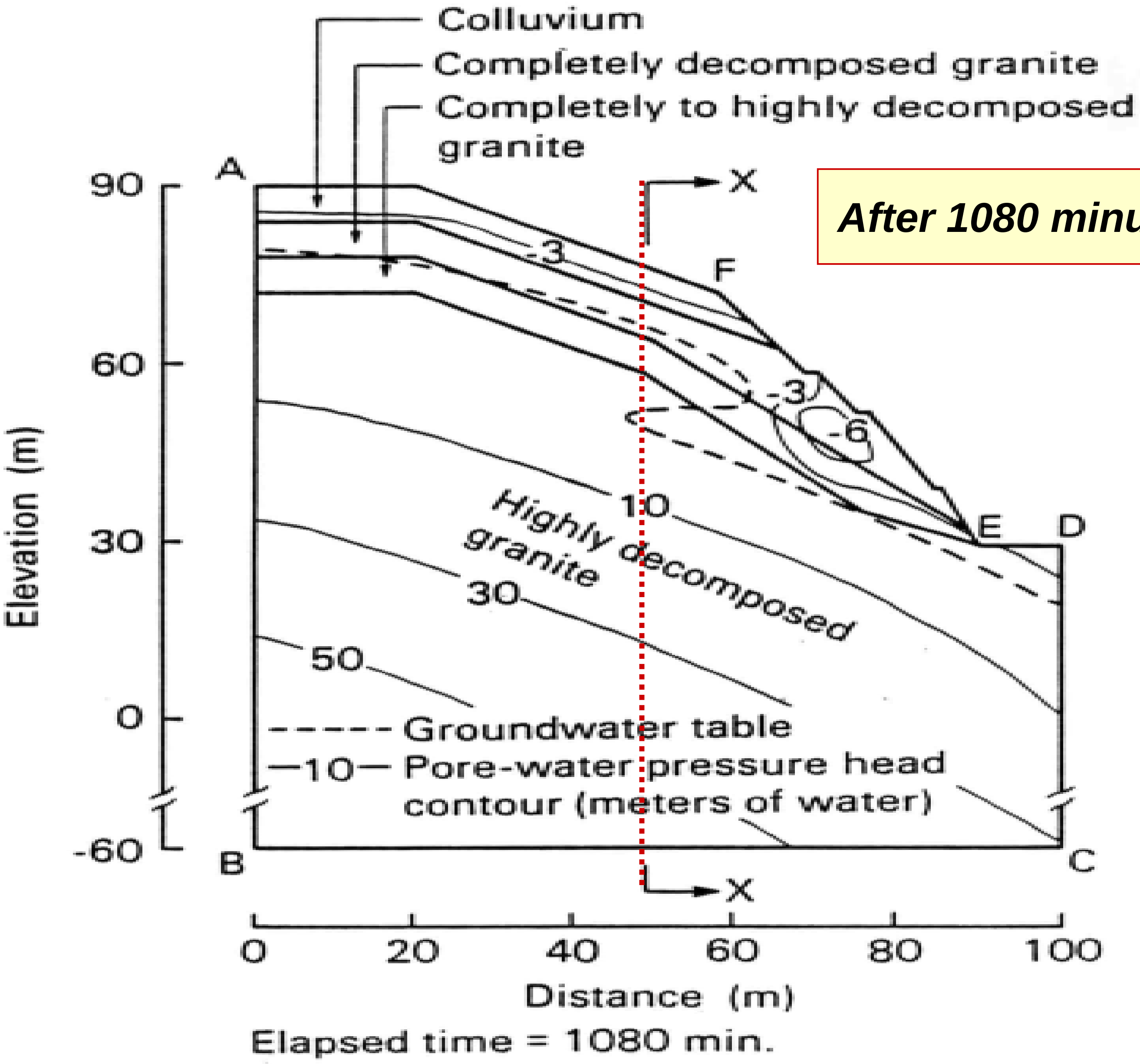
Elapsed time = 120 min.

**After 480 minutes**
**When the rain stops**

**Flux = 0.0**



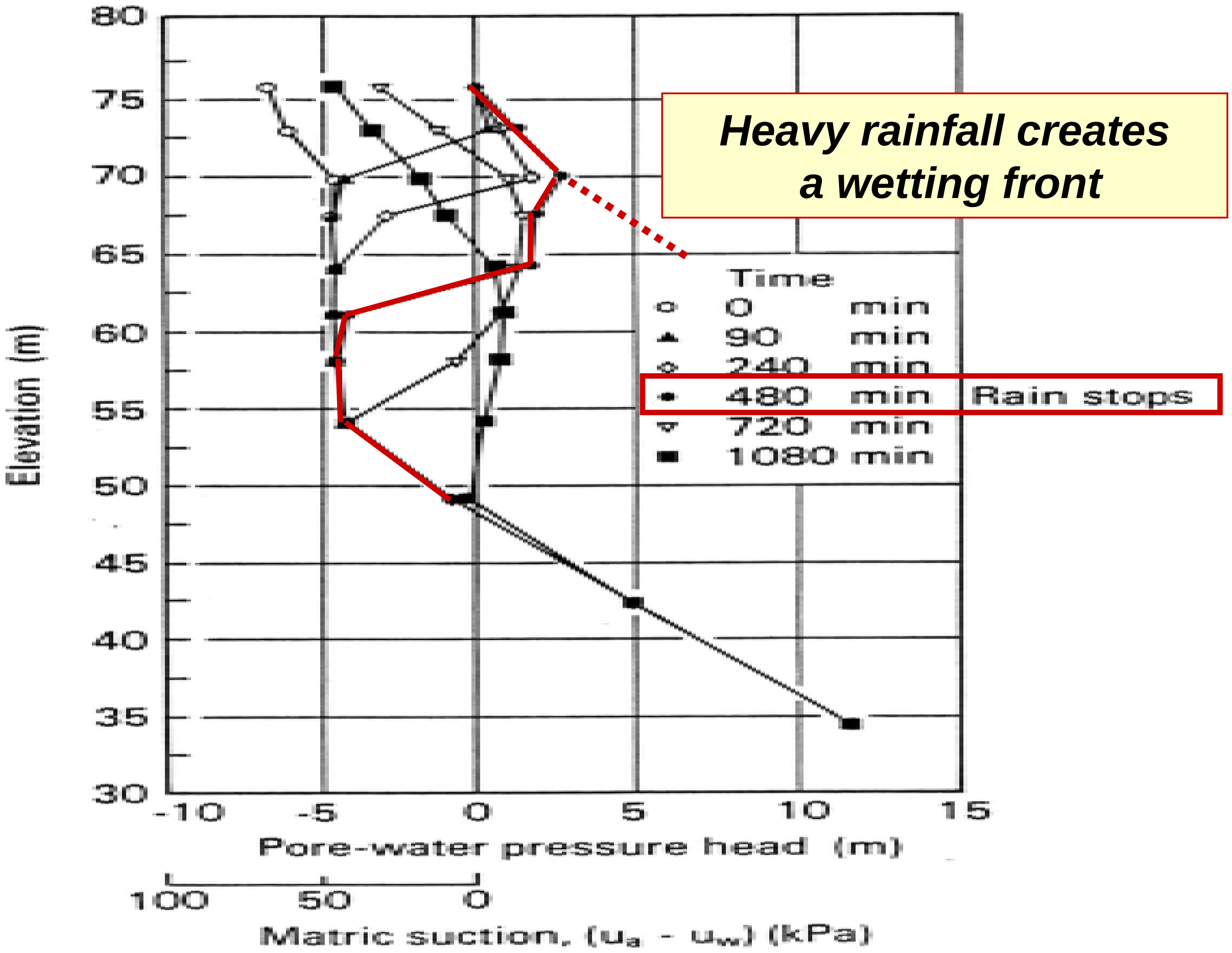
Elapsed time = 480 min.



Elapsed time = 1080 min.

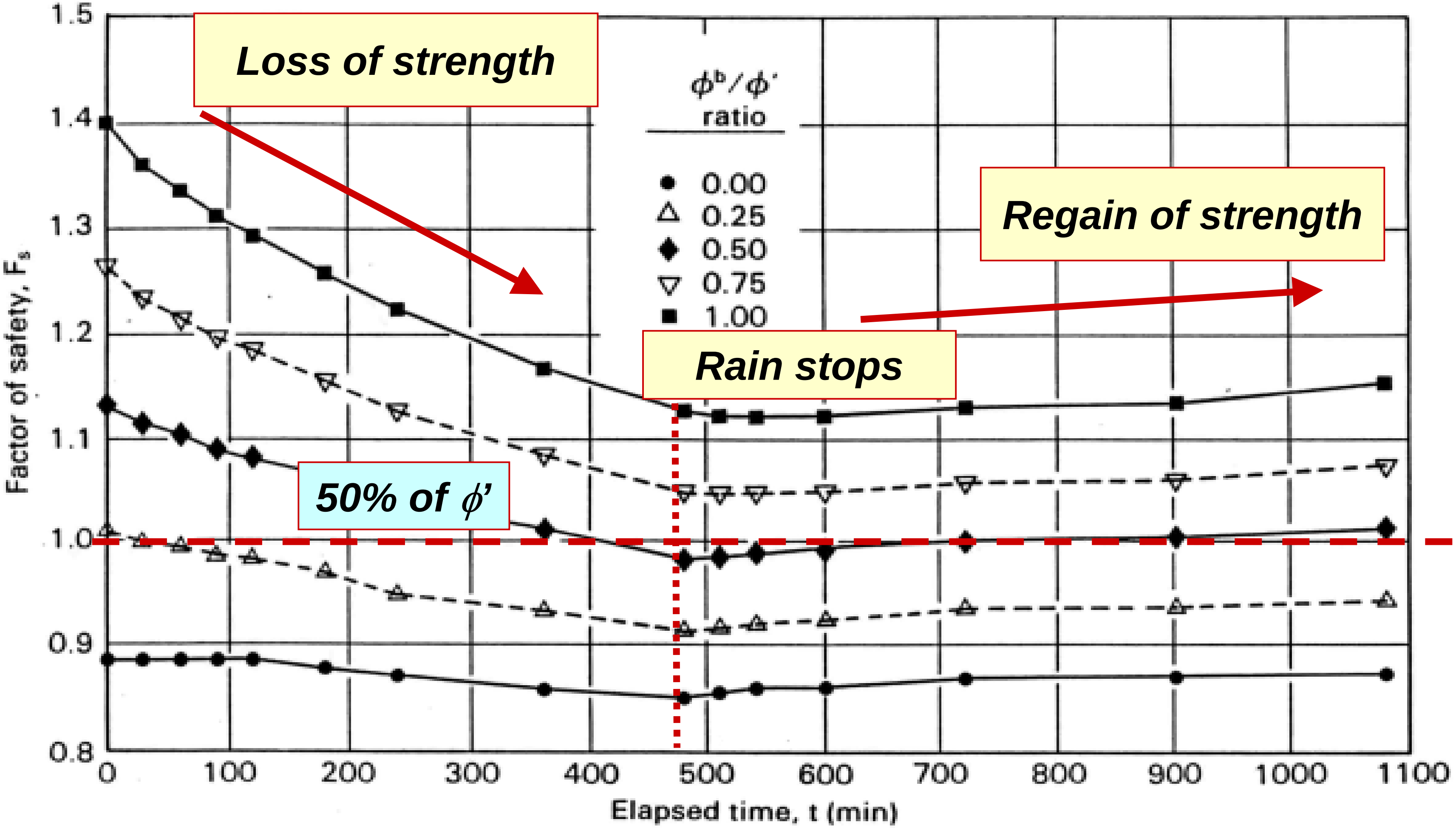
**After 1080 minutes**

**Unsaturated Soil Technology**



**Heavy rainfall creates a wetting front**

Matric suction profiles for section X-X at various elapsed times



**Loss of strength**

**Regain of strength**

**Rain stops**

**50% of $\phi'$**

Factors of safety with respect to elapsed time from the beginning of rainfall