CHAPTER 7 STEADY STATE FLOW

One-dimensional water flow

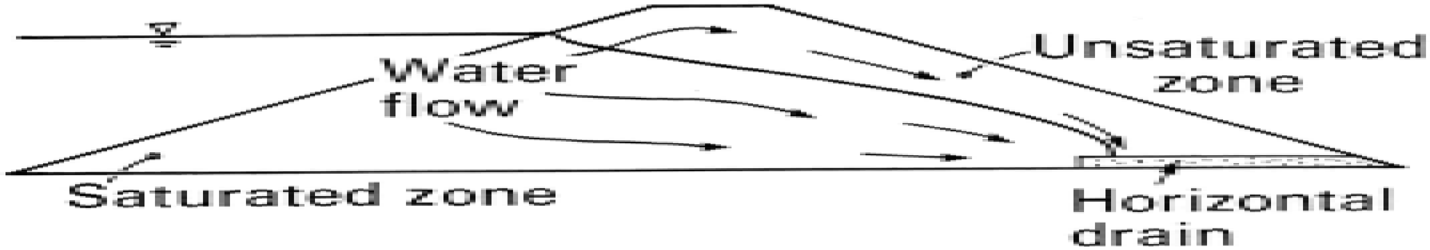
Two-dimensional water flow

3-D water flow

Air flow

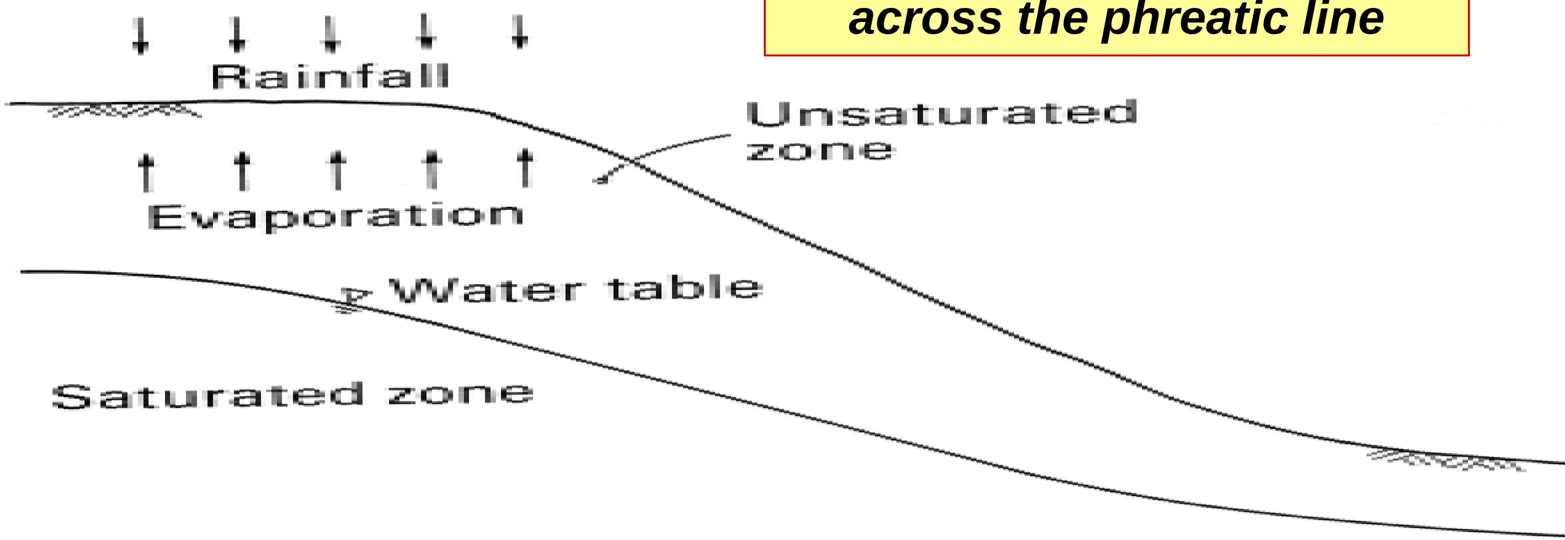
Diffusion

STEADY STATE FLOW



(a) Water flow in the saturated and unsaturated zones of an earth dam

Note that water can flow across the phreatic line

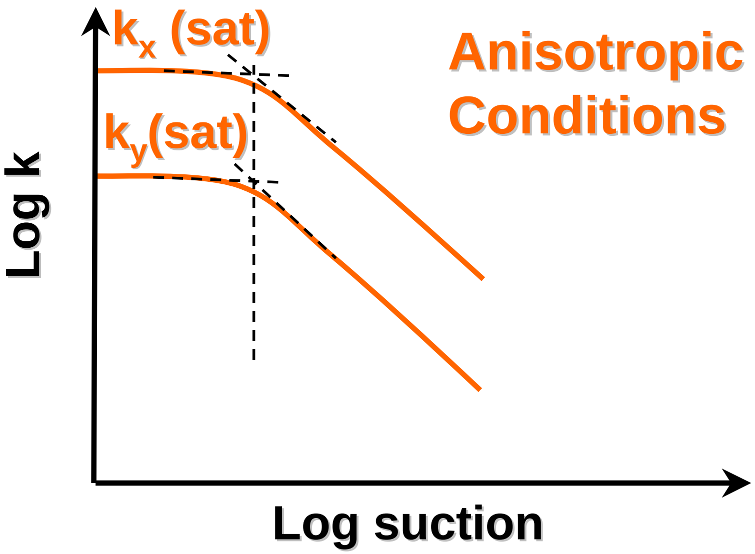


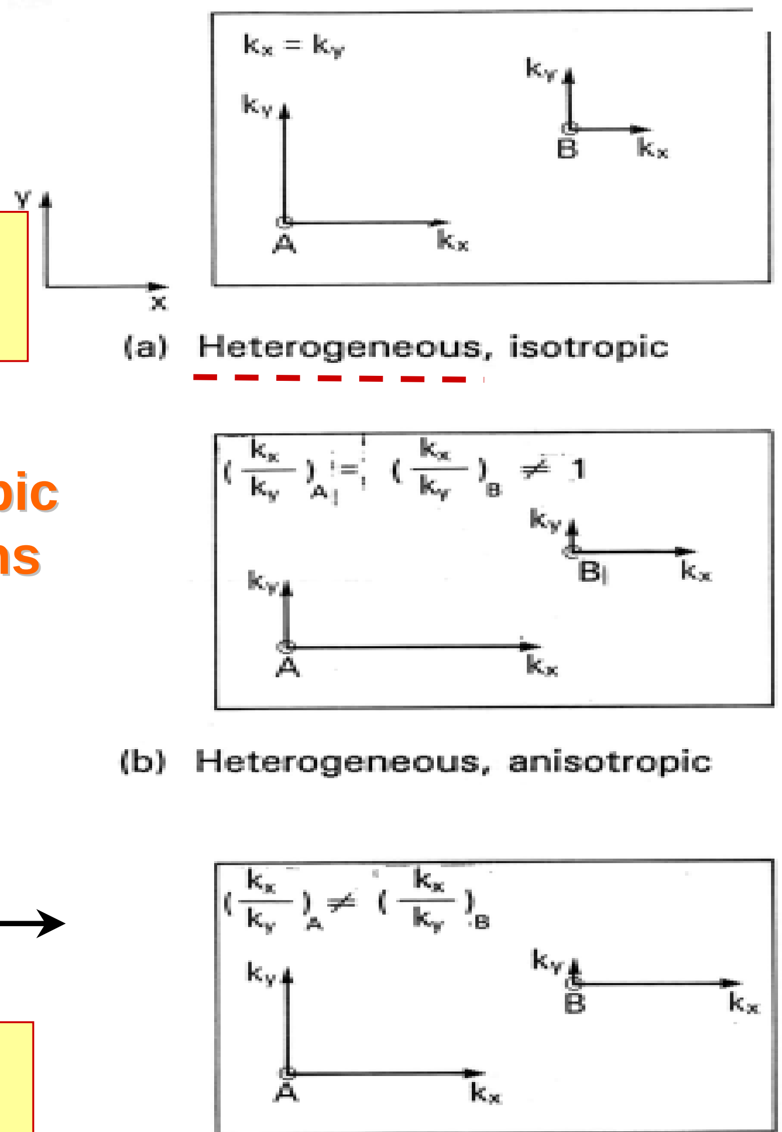
(b) Water flow across the boundary of a slope

Examples involving flow through unsaturated soils

Permeability systems apply to saturated and unsaturated soil systems

Simplest case: Homogeneous, Isotropic



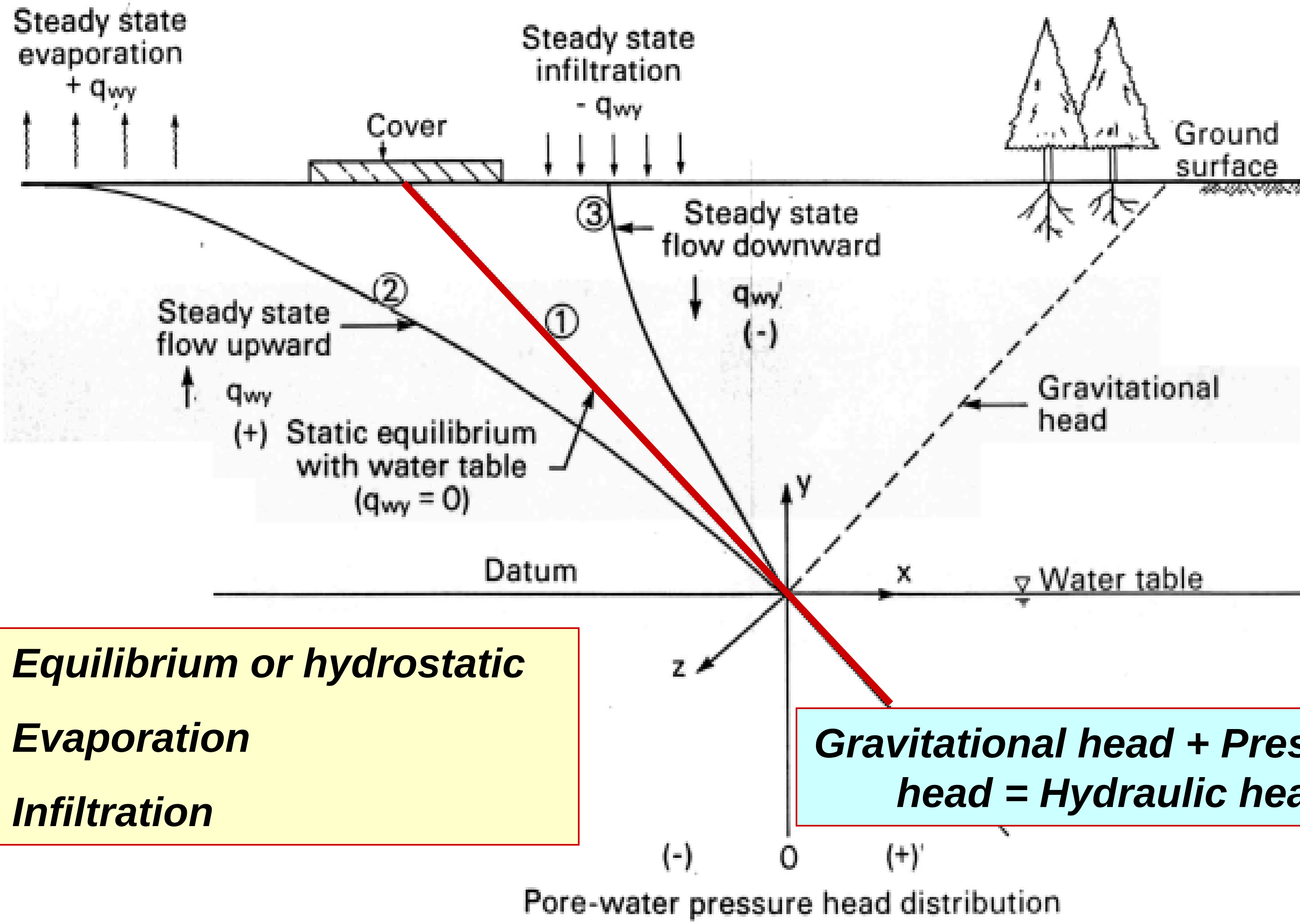


(a) Heterogeneous, isotropic

(b) Heterogeneous, anisotropic

(c) Continuous variation in permeability with space

Some variations in Coefficient of Permeability Systems



1. Equilibrium or hydrostatic

2. Evaporation

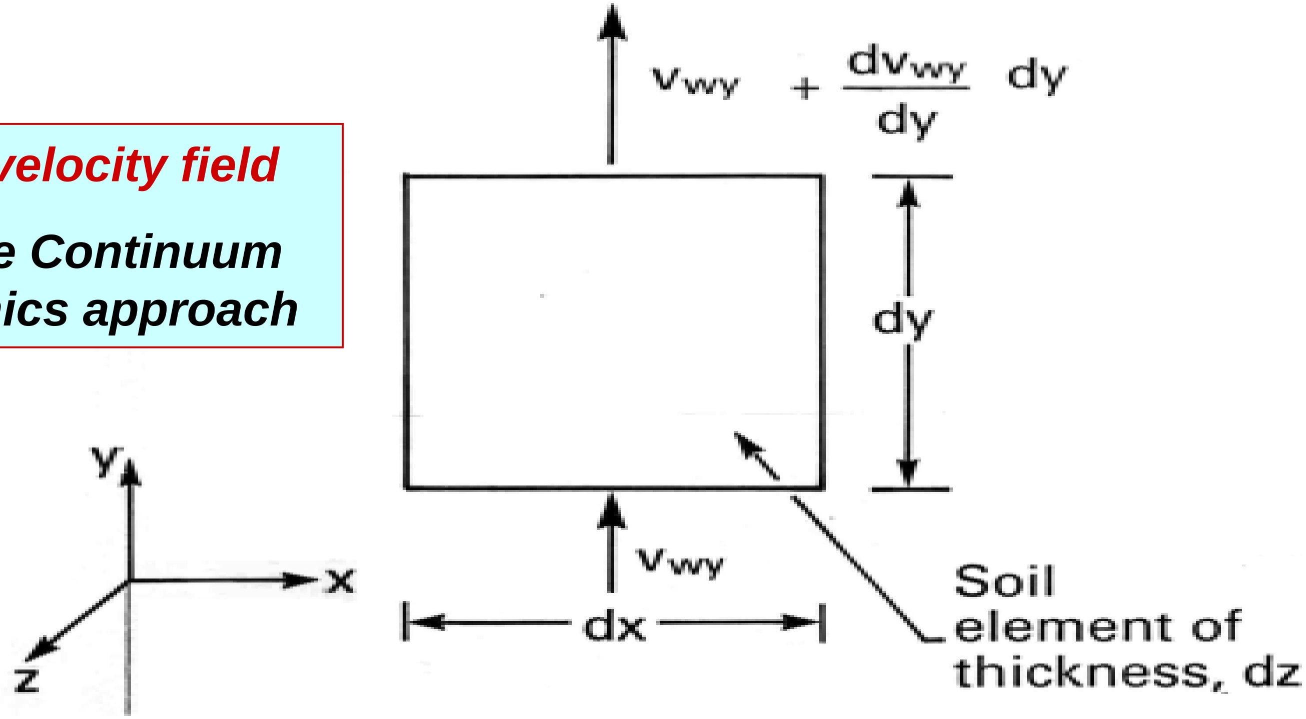
3. Infiltration

Gravitational head + Pressure head = Hydraulic head

Static equilibrium and steady state flow conditions in the zone of negative pore-water pressures

Assume a velocity field

Basic to the Continuum Mechanics approach



Start with Conservation of Mass for the water phase; end up with a CONTINUITY differential equation

One-dimensional water flow through an unsaturated soil element

STEADY STATE WATER FLOW

Formulation for one-dimensional flow

$$\left(v_{wy} + \frac{dv_{wy}}{dy}\,dy\right) dx\,dz - v_{wy}\,dx\,dz = 0$$

Net water flow through an element

where:

v_{wy} = water flow rate across a unit area of the soil in the y-direction

dx, dy, dz = dimensions in the x-, y- and z-directions, respectively

$$\frac{dv_{wy}}{dy}\,dx\,dy\,dz = 0$$

Add the Darcy constitutive law

$$\frac{d\,\{-k_{wy}(u_a - u_w)\,dh_w / dy\}}{dy}\,dx\,dy\,dz = 0$$

where:

$k_{wy}(u_a - u_w)$ = water coefficient of permeability as a function of matric suction which varies with location in the y-direction

dh_w / dy = hydraulic head gradient in the y -direction

h_w = hydraulic head (i.e., gravitational head plus pore-water pressure head)

New component to water flow

$$k_{wy}\frac{d^2h_w}{dy^2} + \frac{dk_{wy}}{dy}\frac{dh_w}{dy} = 0$$

Chain Rule of Differentiation when Deriving the Partial Differential Equation for Saturated/Unsaturated Seepage

$$d\,v_{wy} / dy = 0$$

Net Flow

Divergence of velocity

Substitute in Darcy's law

$$\frac{d\,(k_w\, dh / dy)}{dy} = 0$$

Apply the Chain Rule of Differentiation

$$k_w \frac{d^2h}{dy^2} + \frac{dk_w}{dy}\frac{dh}{dy} = 0$$

Saturated-Unsaturated Flow

Change in permeability

Taylor's Series can be used to compute points along a curved function

$$d^2h_w / dy^2 = 0.0$$

Finite difference method

$$h_{i+1} = h_i + \Delta y \left[\frac{dh}{dy}\right]_i + \frac{\Delta y^2}{2!}\left[\frac{d^2h}{dy^2}\right]_i + \frac{\Delta y^3}{3!}\left[\frac{d^3h}{dy^3}\right]_i + \ldots$$

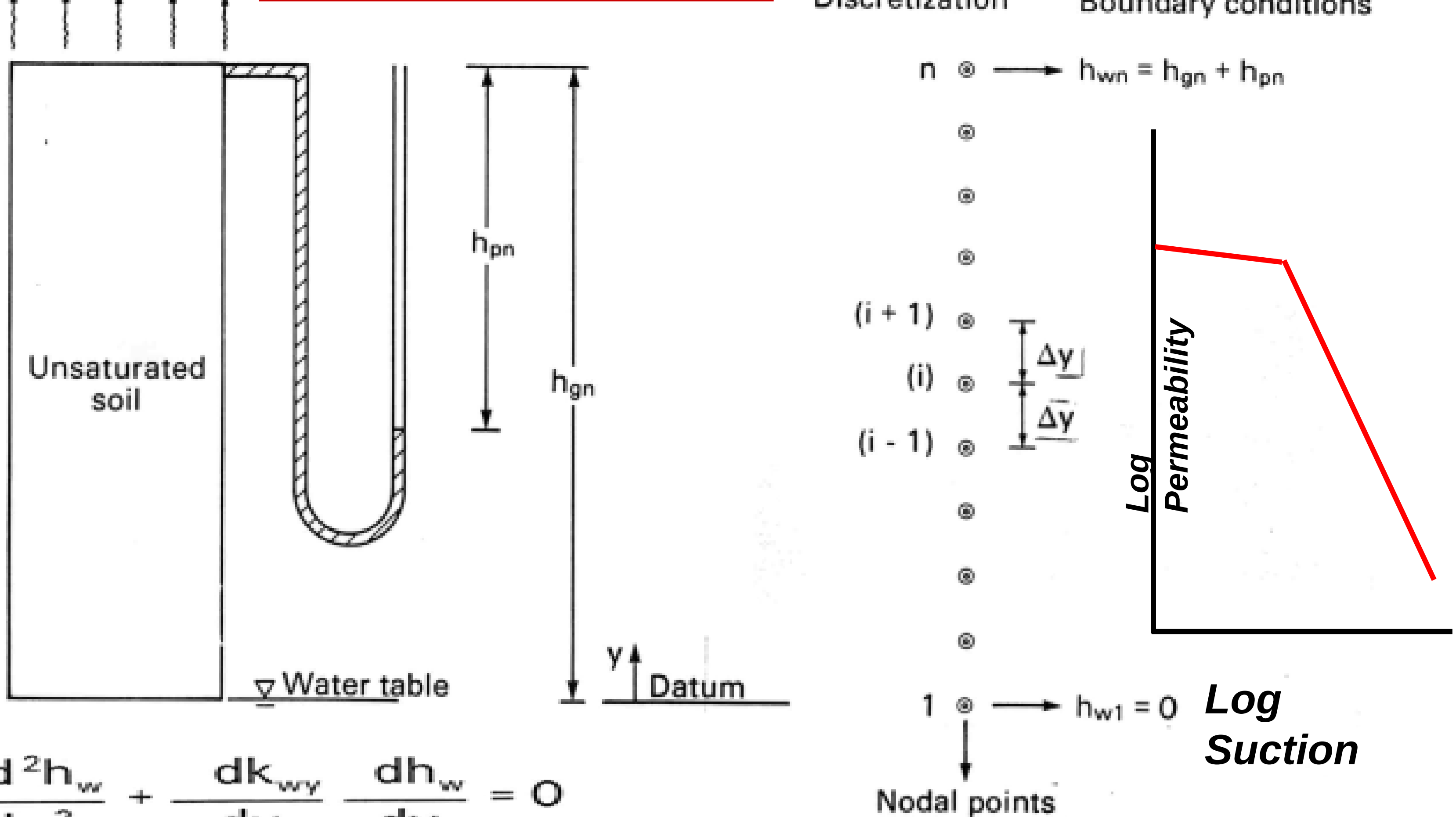
$$h_{i-1} = h_i - \Delta y \left[\frac{dh}{dy}\right]_i + \frac{\Delta y^2}{2!}\left[\frac{d^2h}{dy^2}\right]_i - \frac{\Delta y^3}{3!}\left[\frac{d^3h}{dy^3}\right]_i + \ldots$$

where:
i-1, i, i+1 = three consecutive points spaced at increments, Δy

$$\left[\frac{dh}{dy}\right] = \frac{h_{i+1} - h_{i-1}}{2\Delta y}$$

$$\left[\frac{d^2h}{dy^2}\right]_i = \frac{h_{i+1} + h_{i-1} - 2h_i}{\Delta y^2}$$

Constant Head at Top



$$k_{wy} \frac{d^2 h_w}{dy^2} + \frac{dk_{wy}}{dy} \frac{dh_w}{dy} = 0$$

One-dimensional, steady state water flow through an *Unsaturated Soil* with a constant head boundary condition

Finite difference method

Head boundary condition

finite difference form for point (i).

$$k_{wy(i)} \left\{ \frac{h_{w(i+1)} + h_{w(i-1)} - 2h_{w(i)}}{(\Delta y)^2} \right\} + \left\{ \frac{k_{wy(i+1)} - k_{wy(i-1)}}{2\Delta y} \right\}$$

$$k_{wy} \frac{d^2 h_w}{dy^2} + \frac{dk_{wy}}{dy} \frac{dh_w}{dy} = 0$$

$$\left\{ \frac{h_{w(i+1)} - h_{w(i+1)}}{2\Delta y} \right\} = 0$$

where:

$k_{wy(i)}, k_{wy(i-1)}, k_{wy(i+1)}$ = water coefficients of permeability in the y-direction at points (i), (i-1) and (i + 1), respectively

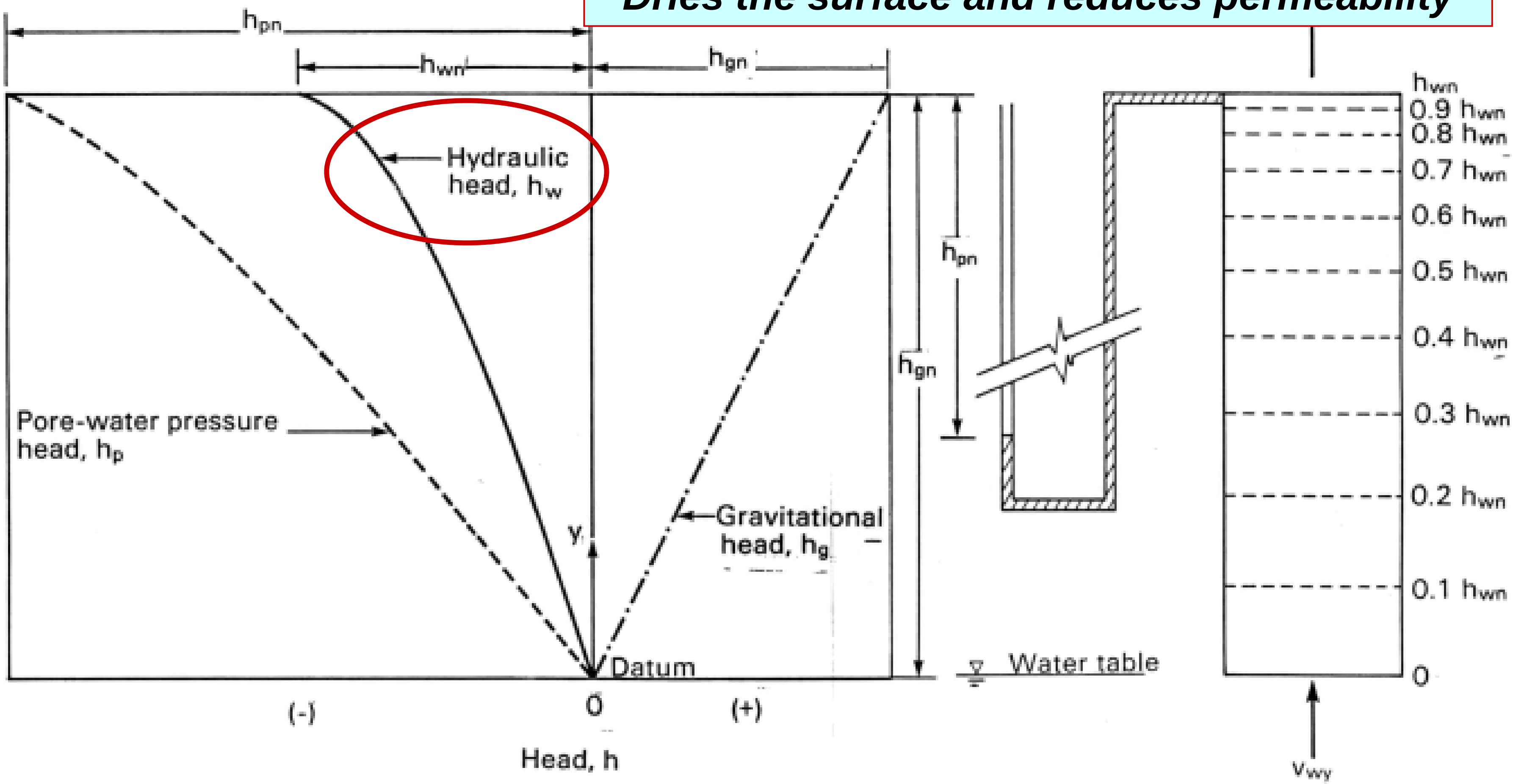
$h_{w(i)}, h_{w(i-1)}, h_{w(i+1)}$ = hydraulic heads at points (i), (i-1) and (i + 1), respectively

$$-\{8\, k_{wy(i)}\}\, h_{w(i)} + \{4\, k_{wy(i)} + k_{wy(i+1)} - k_{wy(i-1)}\}\, h_{w(i+1)}$$
$$+ \{4\, k_{wy(i)} + k_{wy(i-1)} - k_{wy(i+1)}\}\, h_{w(i-1)} = 0$$

$h_{w(1)} = 0.0$ at y equal to 0.0 (base)

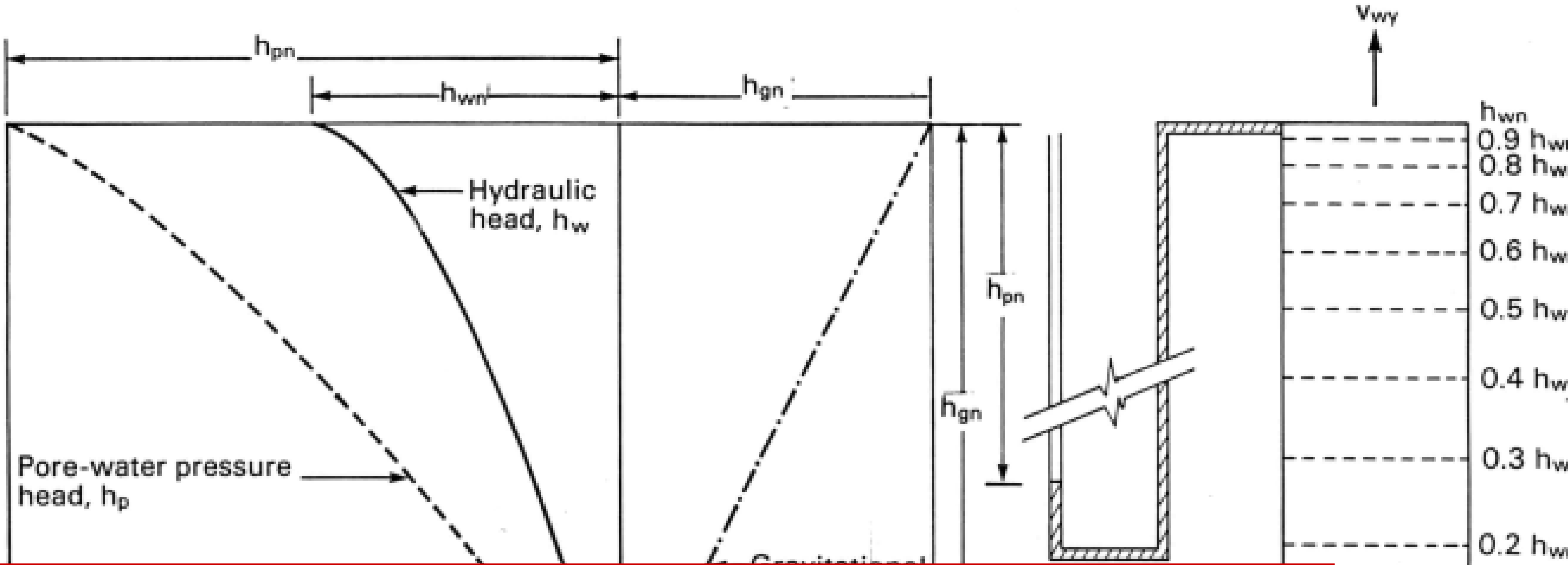
$h_{w(n)} = h_{gn} + h_{pn}$ at y equal to h_{gn} (top)

Dries the surface and reduces permeability



Steady state evaporation through an unsaturated soil column

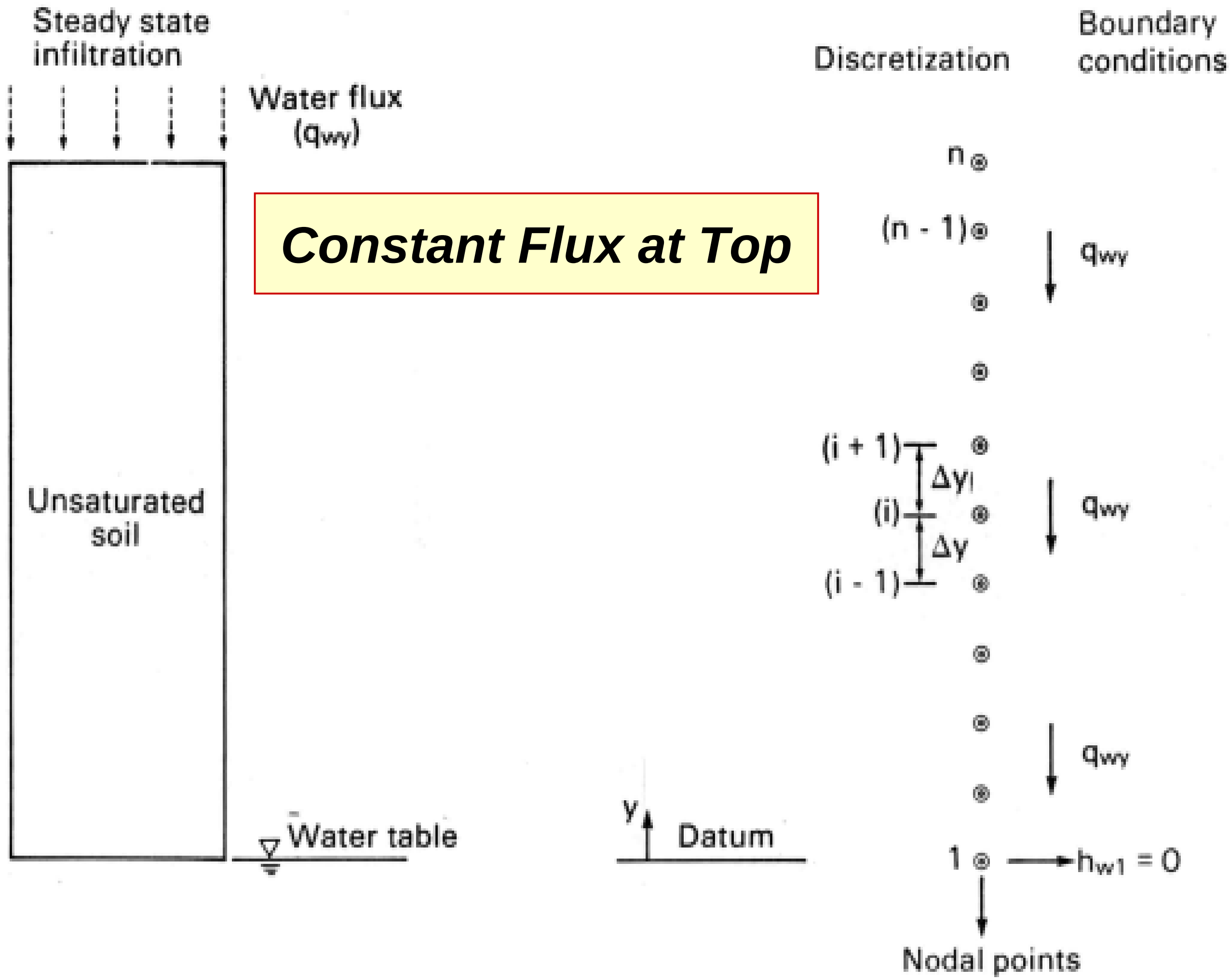
Nonlinearity in pressure head due to nonlinearity in the coefficient of permeability



Procedure for Solving the Partial Differential Seepage Equation

1. Assume the soil has a coefficient of permeability equal to k_{sat}
2. Calculate the heads at all nodes
3. Compute the pore-water pressures from the heads
4. Determine new coefficients of permeability based on new u_w
5. Solve the partial differential equation for new heads
6. Repeat the process until there is NO CHANGE in heads or k
7. Then solution has converged!!!

0.1 h_{wn}
0
v_{wy}



One-dimensional steady state water flow through an unsaturated soil with a flux boundary condition

Finite difference method

Flux boundary condition

Use Darcy's law to relate flux and heads

$$q_{wy} = -k_{wy(i)} \frac{h_{w(i+1)} - h_{w(i-1)}}{2\Delta y} A$$

q_{wy} = water flux through the soil column during the steady state flow. The flux is assumed positive in an upward direction and negative in a downward direction

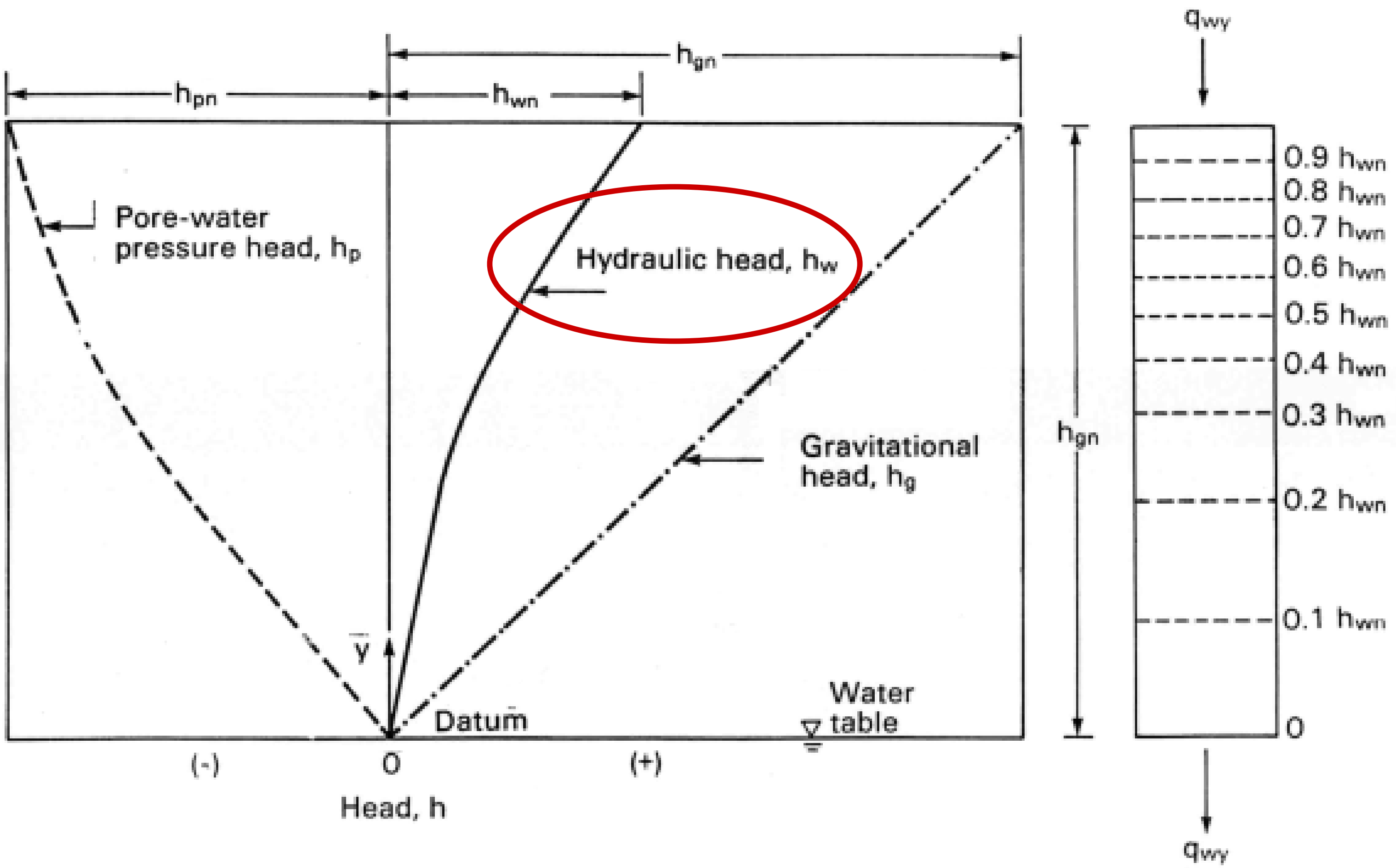
A = cross-sectional area of the soil column

$$h_{w(i+1)} = h_{w(i-1)} - \frac{2\Delta y}{A\, k_{wy(i)}} q_{wy}$$

$$-\{8k_{wy(i)}\} h_{w(i)} + \{4k_{wy(i)} + k_{wy(i+1)} - k_{wy(i-1)}\}\{h_{w(i-1)} - \frac{2\Delta y}{A\, k_{wy(i)}} q$$

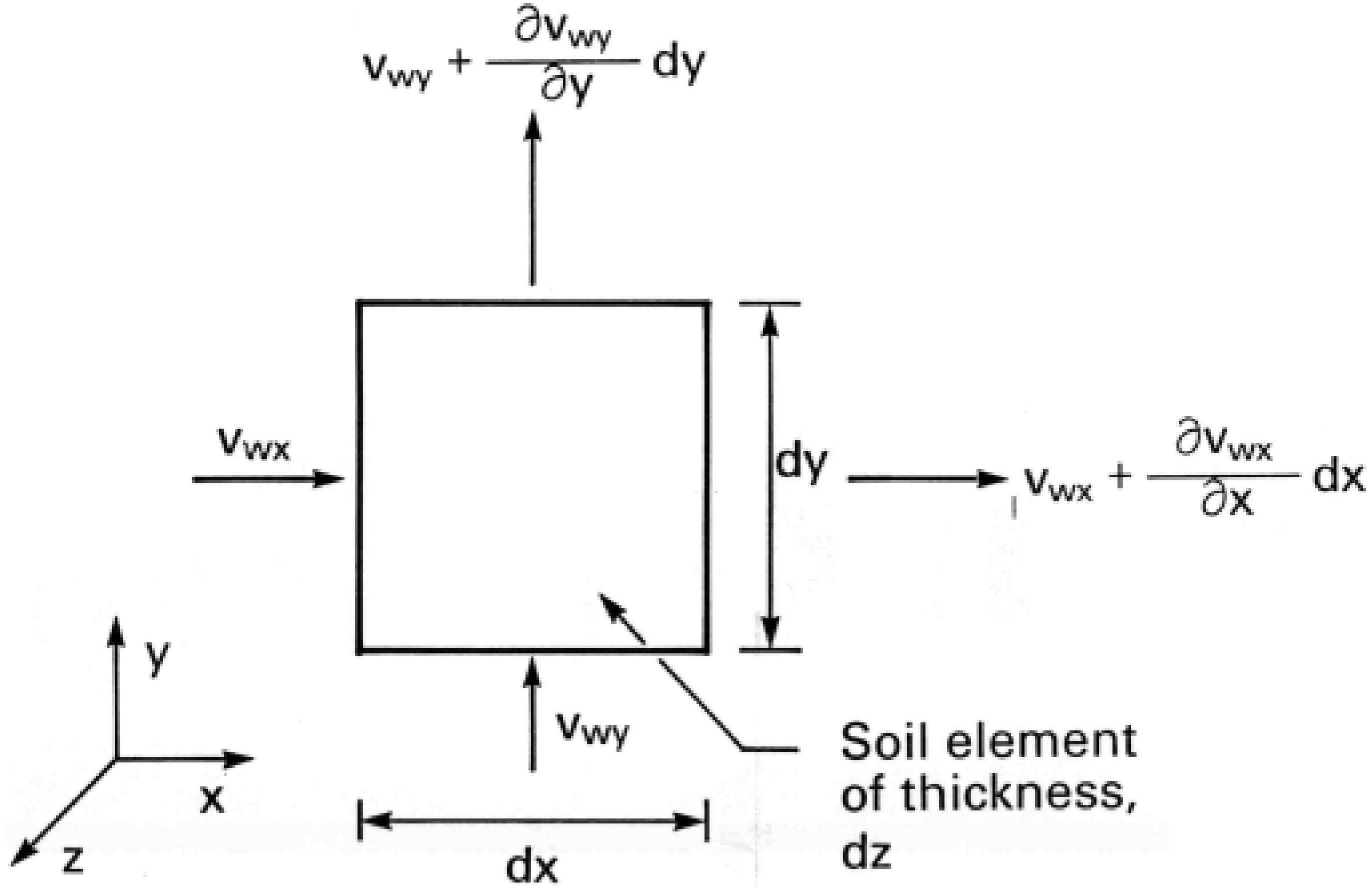
$$\{4k_{wy(i)} + k_{wy(i-1)} - k_{wy(i+1)}\} h_{w(i-1)} = 0$$

$$h_{w(i)} = h_{w(i-1)} - \left\{ \frac{4k_{wy(i)} + k_{wy(i+1)} - k_{wy(i-1)}}{8k_{wy(i)}^2} \right\} \frac{2\Delta y}{A} q_{wy}$$



Steady state infiltration through an unsaturated soil

Nonlinearity in pressure head due to nonlinearity in the coefficient of permeability



Two-dimensional water flow through an unsaturated soil element

Formulation for two-dimensional flow

$$\left(v_{wx} + \frac{\partial v_{wx}}{\partial x} dx - v_{wx}\right) dy\, dz + \left(v_{wy} + \frac{\partial v_{wy}}{\partial y} dy - v_{wy}\right) dx\, dz = 0$$

where:

v_{wx} = water flow rate across a unit area of the soil in the x-direction

Therefore,

Divergence of velocity

$$\left[\frac{\partial v_{wx}}{\partial x} + \frac{\partial v_{wy}}{\partial y}\right] dx\, dy\, dz = 0$$

where: $v_{wx} = k_{wx}\, dh_w / dx$

$v_{wy} = k_{wy}\, dh_w / dy$

where:

$k_{wx}(u_a - u_w)$ = water coefficients of permeability as a function of matric suction. The permeability can vary with location in the x-direction

$\partial h_w / \partial x$ = hydraulic head gradient in the x -direction

$$k_{wx} \frac{\partial^2 h_w}{\partial x^2} + k_{wy} \frac{\partial^2 h_w}{\partial y^2} + \frac{\partial k_{wx}}{\partial x} \frac{\partial h_w}{\partial x} + \frac{\partial k_{wy}}{\partial y} \frac{\partial h_w}{\partial y} = 0$$

where:

$\partial k_{wx} / \partial x$ = change in water coefficient of permeability in the x-direction

Two-Dimensional Steady State Equations for Unsaturated Soils

Heterogeneous, anisotropic

$$k_{wx}\frac{\partial^2 h_w}{\partial x^2} + k_{wy}\frac{\partial^2 h_w}{\partial y^2} + \frac{\partial k_{wx}}{\partial x}\frac{\partial h_w}{\partial x} + \frac{\partial k_{wy}}{\partial y}\frac{\partial h_w}{\partial y} = 0$$

Heterogeneous, isotropic

$$k_w\left[\frac{\partial^2 h_w}{\partial x^2} + \frac{\partial^2 h_w}{\partial y^2}\right] + \frac{\partial k_w}{\partial x}\frac{\partial h_w}{\partial x} + \frac{\partial k_w}{\partial y}\frac{\partial h_w}{\partial y} = 0$$

An unsaturated soil is a heterogeneous soil since permeability varies with space

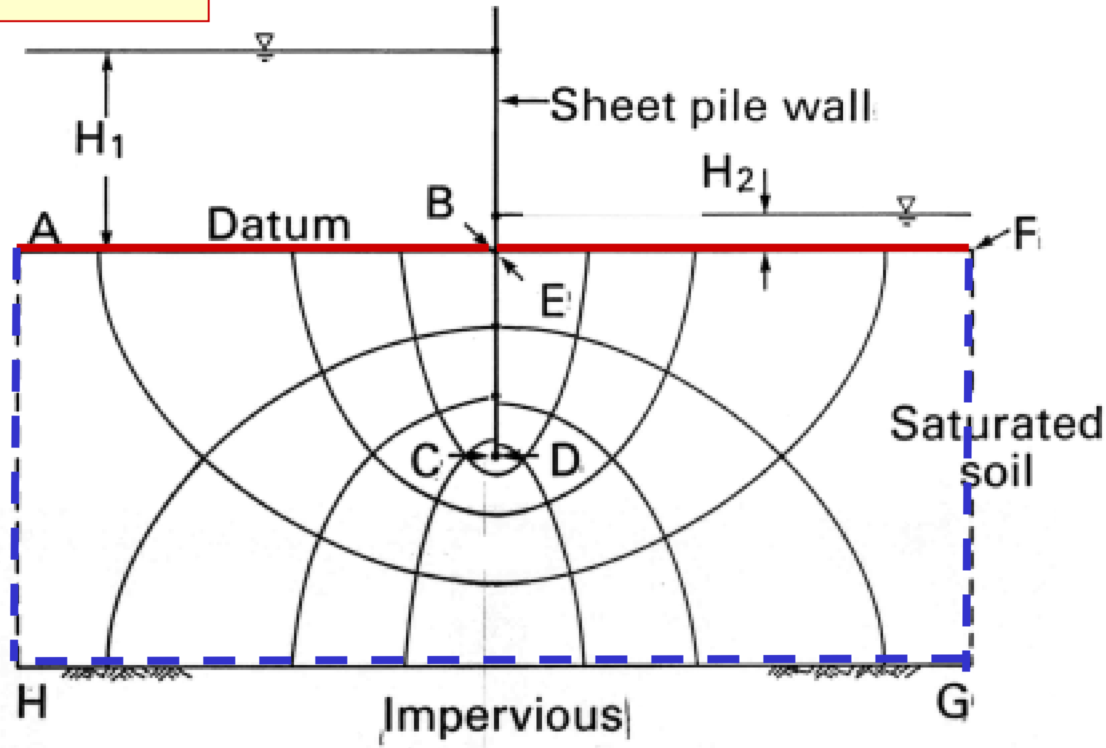
Two-Dimensional Steady State Equations for Saturated Soils

Anisotropic	Isotropic
Heterogeneous	
$k_{sx} \frac{\partial^2 h_w}{\partial x^2} + k_{sy} \frac{\partial^2 h_w}{\partial y^2} + \frac{\partial k_{sx}}{\partial x} \frac{\partial h_w}{\partial x} + \frac{\partial k_{sy}}{\partial y} \frac{\partial h_w}{\partial y} = 0$	$k_s \left[\frac{\partial^2 h_w}{\partial x^2} + \frac{\partial^2 h_w}{\partial y^2} \right] + \frac{\partial k_s}{\partial x} \frac{\partial h_w}{\partial x} + \frac{\partial k_s}{\partial y} \frac{\partial h_w}{\partial y} = 0$
Homogeneous	
$k_{sx} \frac{\partial^2 h_w}{\partial x^2} + k_{sy} \frac{\partial^2 h_w}{\partial y^2} = 0$	$\frac{\partial^2 h_w}{\partial x^2} + \frac{\partial^2 h_w}{\partial y^2} = 0$

Also applies for an unsaturated soil

LaPlace partial differential equation that can be solved using the flownet technique

Boundary Conditions



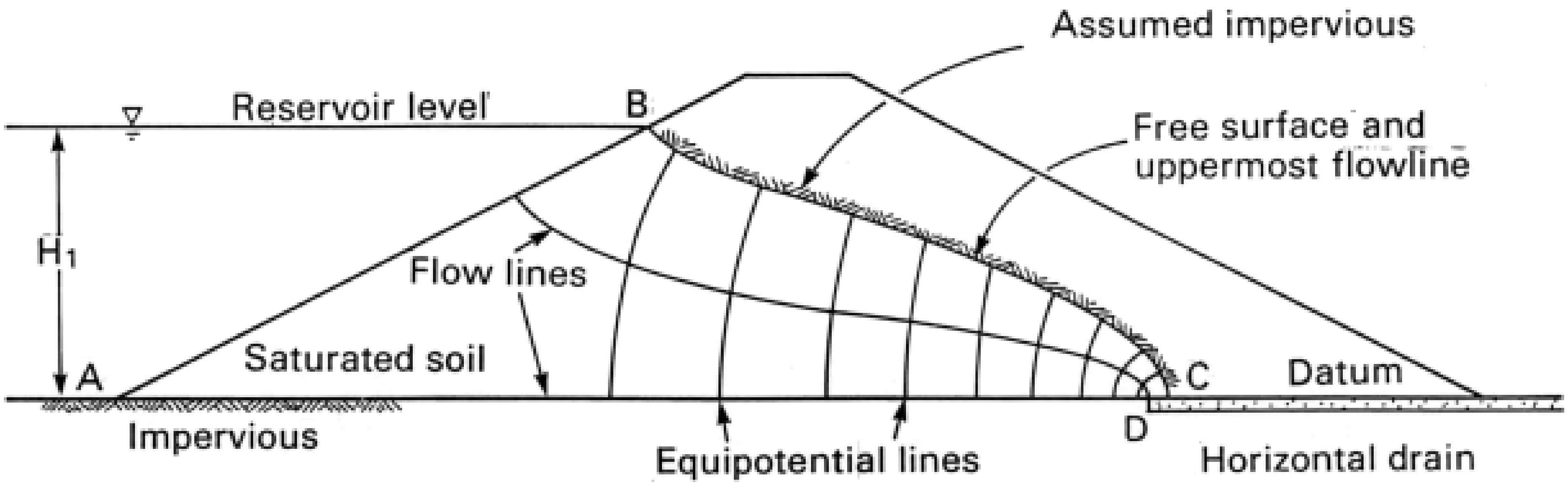
Boundary conditions:
AB: $h_w = H_1$
BC and DE: $q_w = 0$
EF: $h_w = H_2$
AH and FG: $q_w = 0$
HG: $q_w = 0$

LaPlace partial differential equation solution using the flownet technique

(a) Steady state seepage throughout a homogeneous, isotropic saturated soil

Unsaturated Soil Technology

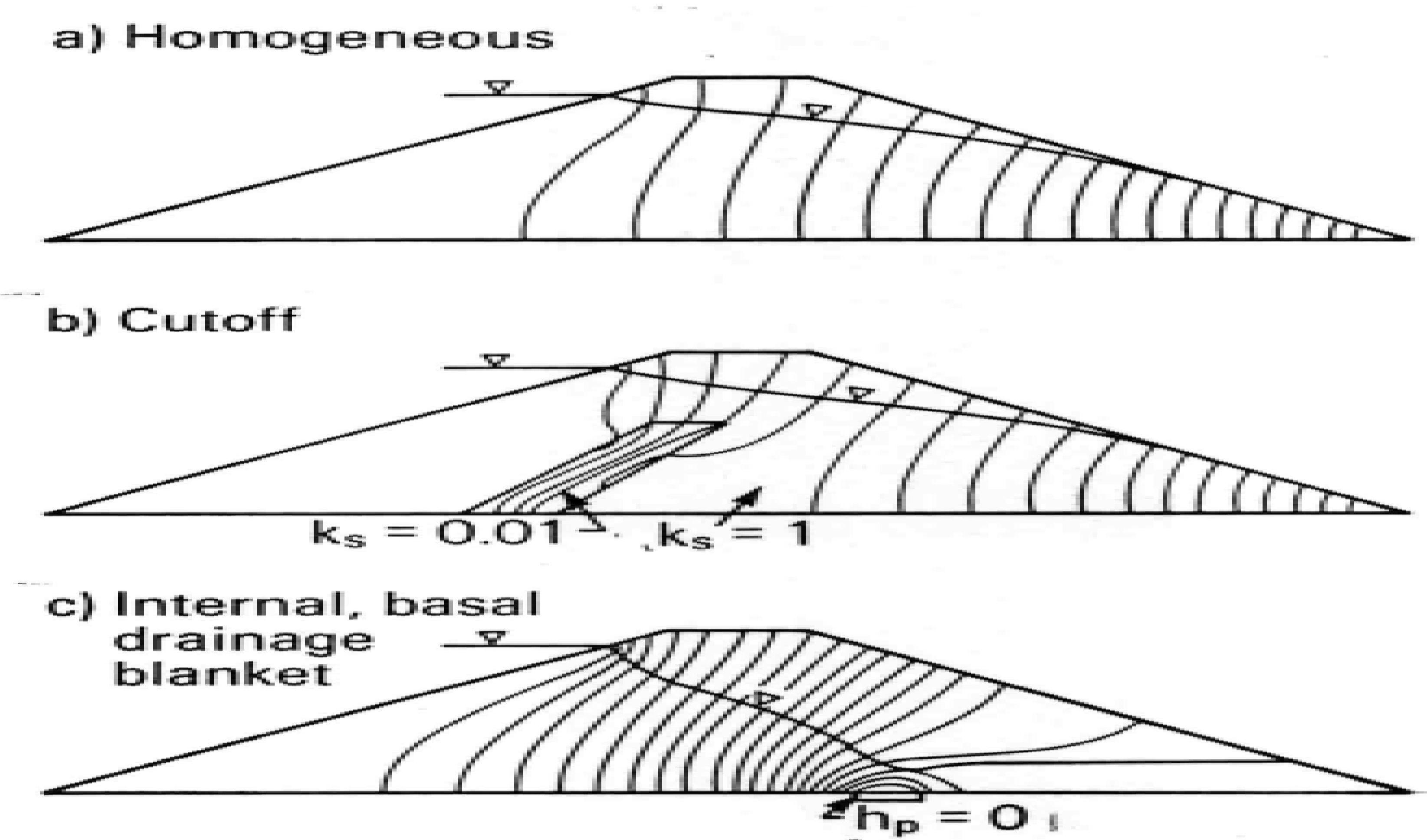
Creation of the Unconfined Category of Seepage Problems



Boundary conditions:
AB: $h_w = H_1$
BC: free surface, it's location is unkown
CD: $h_w = 0$
DA: $q_w = 0$

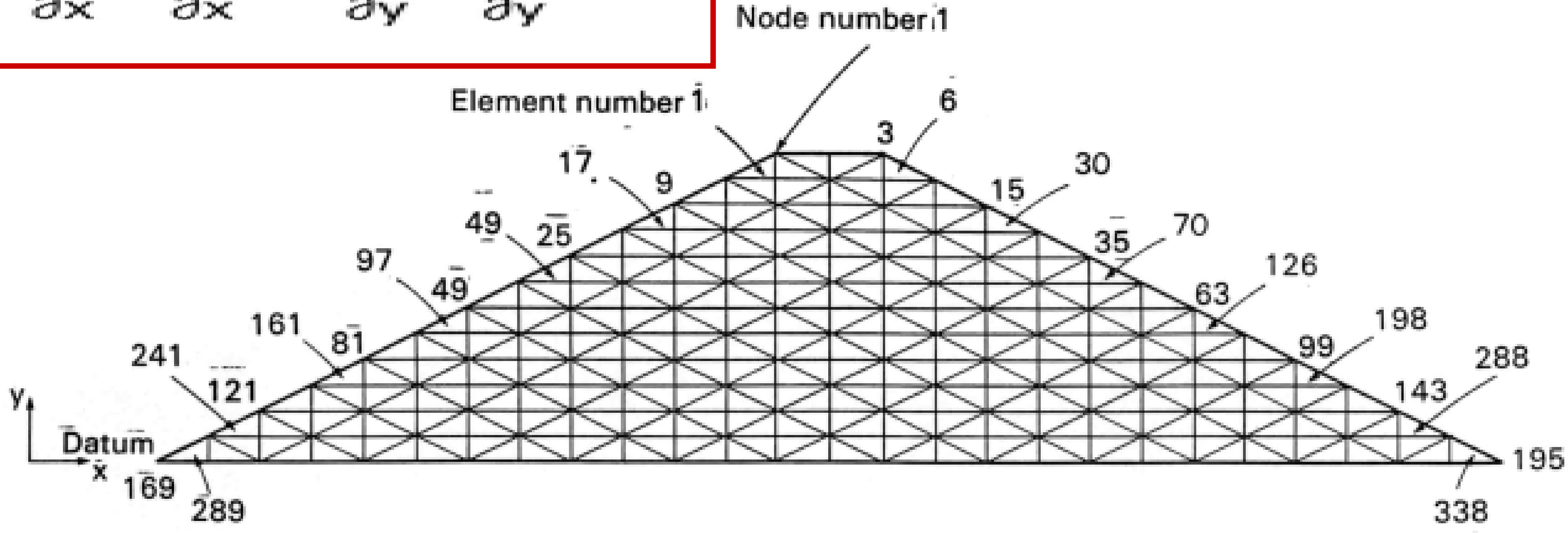
(b) Steady state seepage throughout a homogeneous, isotropic earth dam

***Problem:* An attempt is being made to impose two boundary conditions at the phreatic surface; no flow and zero pressure**



Examples of saturated-unsaturated seepage modeling (Freeze, 1971)

$$k_{sx}\frac{\partial^2 h_w}{\partial x^2} + k_{sy}\frac{\partial^2 h_w}{\partial y^2} + \frac{\partial k_{sx}}{\partial x}\frac{\partial h_w}{\partial x} + \frac{\partial k_{sy}}{\partial y}\frac{\partial h_w}{\partial y} = 0$$



Discretized cross-section of a dam for a finite element analysis

Examples of saturated-unsaturated seepage modeling (Papagianakis and Fredlund, 1984)

Unsaturated Soil Technology

Solutions for two-dimensional flow
Seepage analysis using the finite element method

$$\int_A \begin{bmatrix} \frac{\partial}{\partial x}\{L\} \\ \frac{\partial}{\partial y}\{L\} \end{bmatrix}^T \begin{bmatrix} k_{wx} & 0 \\ 0 & k_{wy} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x}\{L\} \\ \frac{\partial}{\partial y}\{L\} \end{bmatrix} dA\,\{h_{wn}\} - \int_S \{L\}^T \bar{v}_w\, dS = 0$$

{L} ***Element area coordinates***

where:

$\{L\}$ = matrix of the element area coordinates (i.e., $\{L_1\ L_2\ L_3\}$)

L_1, L_2, L_3 = area coordinates of points in the element that are related to the cartesian coordinates of nodal points

$L_1 = 1/2A\{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$

$L_2 = 1/2A\{(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\}$

$L_3 = 1/2A\{(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\}$

$x_i, y_i (i = 1,2,3)$ = cartesian coordinates of the three nodal points of an element

x, y = cartesian coordinates of a point within the element

A = area of the element

Galerkin principle of Weighted Residuals

[C] ***Constitutive matrix***

$\begin{bmatrix} k_{wx} & 0 \\ 0 & k_{wy} \end{bmatrix}$ = matrix of the water coefficients of permeability (i.e., $[k_w]$)

$\{h_{wn}\}$ = matrix of hydraulic heads at the nodal points $\begin{Bmatrix} h_{w1} \\ h_{w2} \\ h_{w3} \end{Bmatrix}$

$\bar{v}_w$ = external water flow rate in a direction perpendicular to the boundary of the element

S = perimeter of the element

$$\int_A [B]^T [k_w] [B] dA \{h_{wn}\} - \int_S [L]^T \bar{v}_w \, dS = 0$$

where:

Simplified Finite Element form for the PDE

$[B]$ = matrix of the derivatives of the area coordinates which can be written as.

$$[B] = \frac{1}{2A} \begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

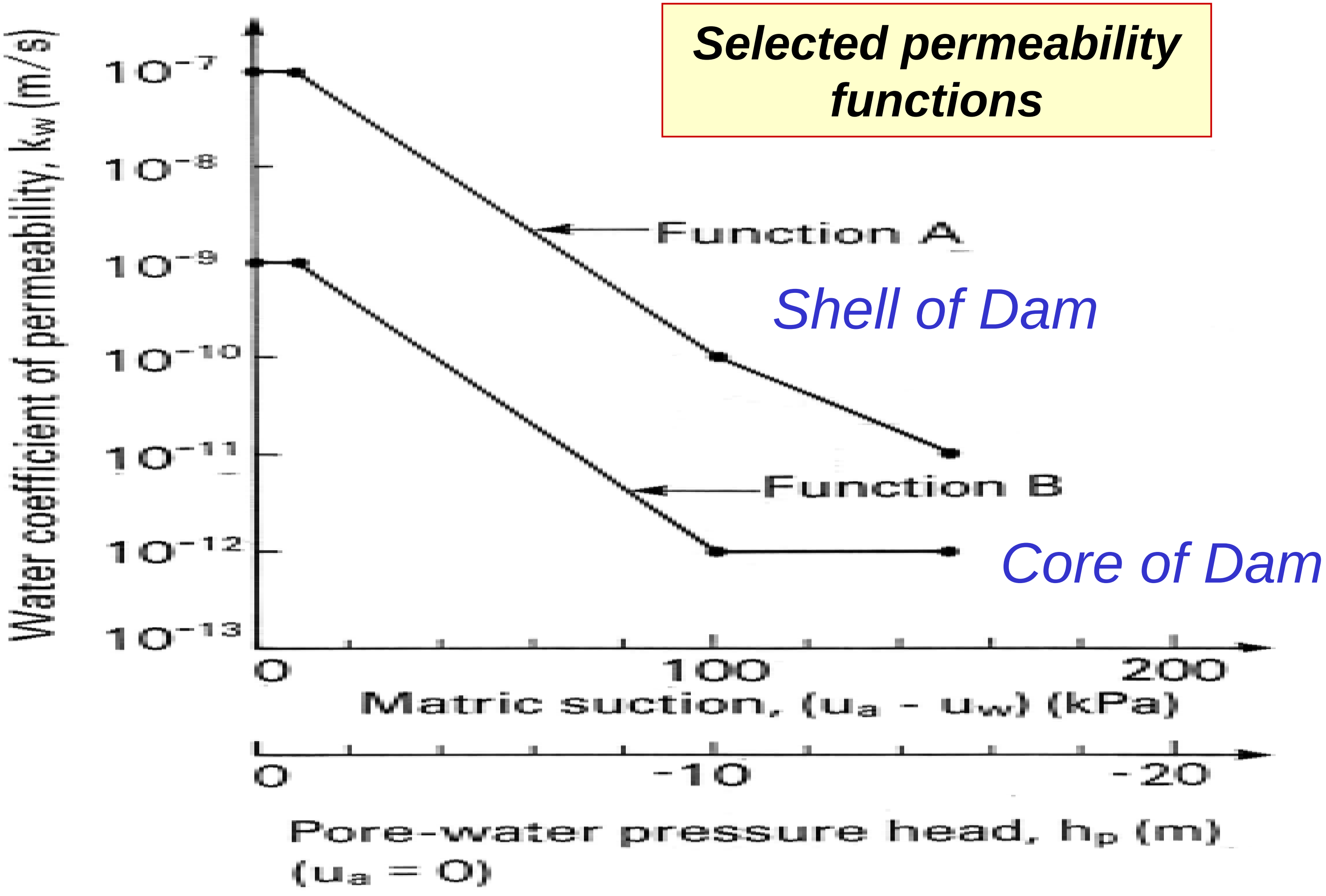
$$\begin{Bmatrix} v_{wx} \\ v_{wy} \end{Bmatrix} = [k_w] [B] \{h_{wn}\}$$

Solve for velocities after heads are calculated

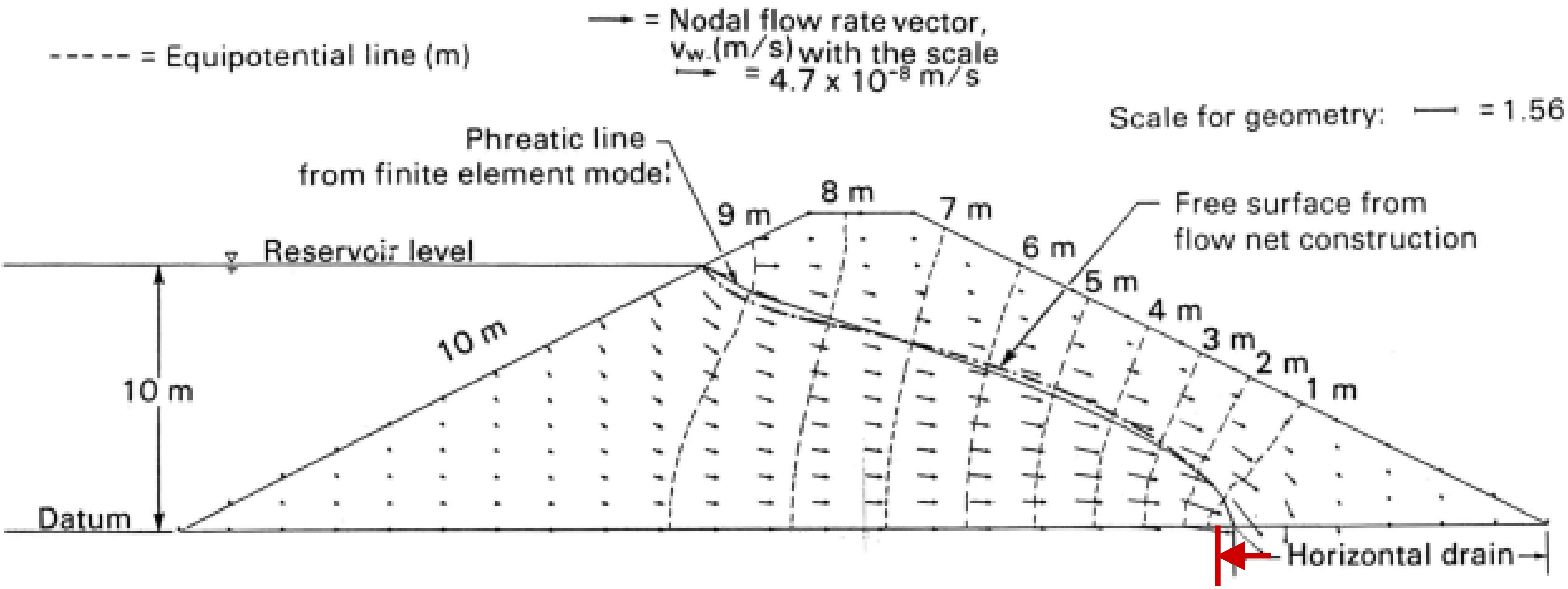
where:

v_{wx}, v_{wy} = water flow rates within an element in the x- and y-directions, respectively

Selected permeability functions

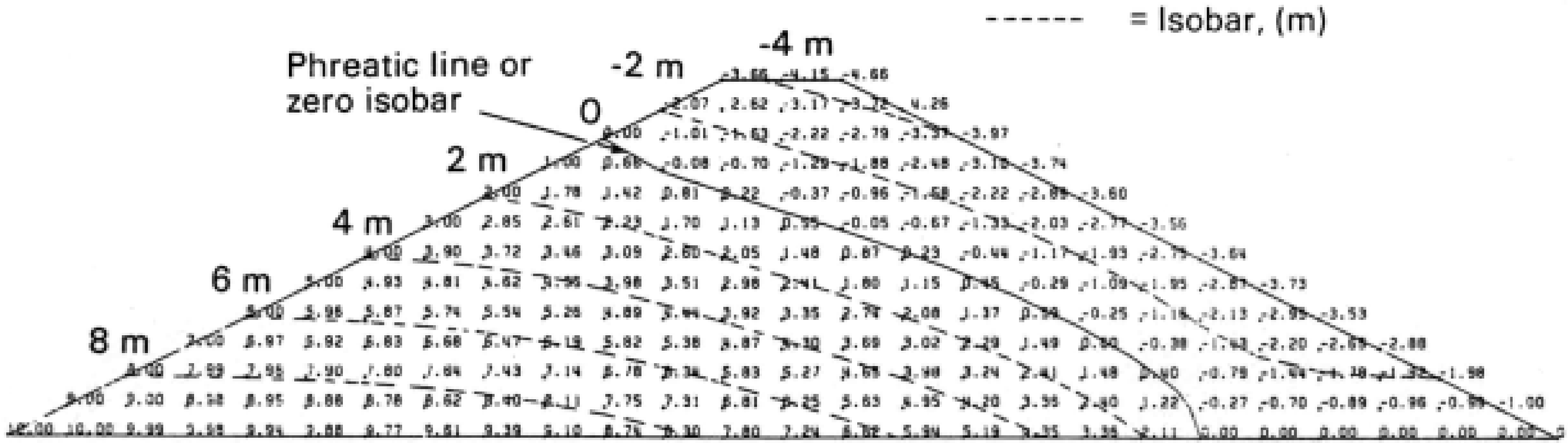


Specified permeability functions for analyzing steady state seepage through a dam



(a) Equipotential lines and nodal flow rate vectors through the dam

Isotropic earth dam with a horizontal drain

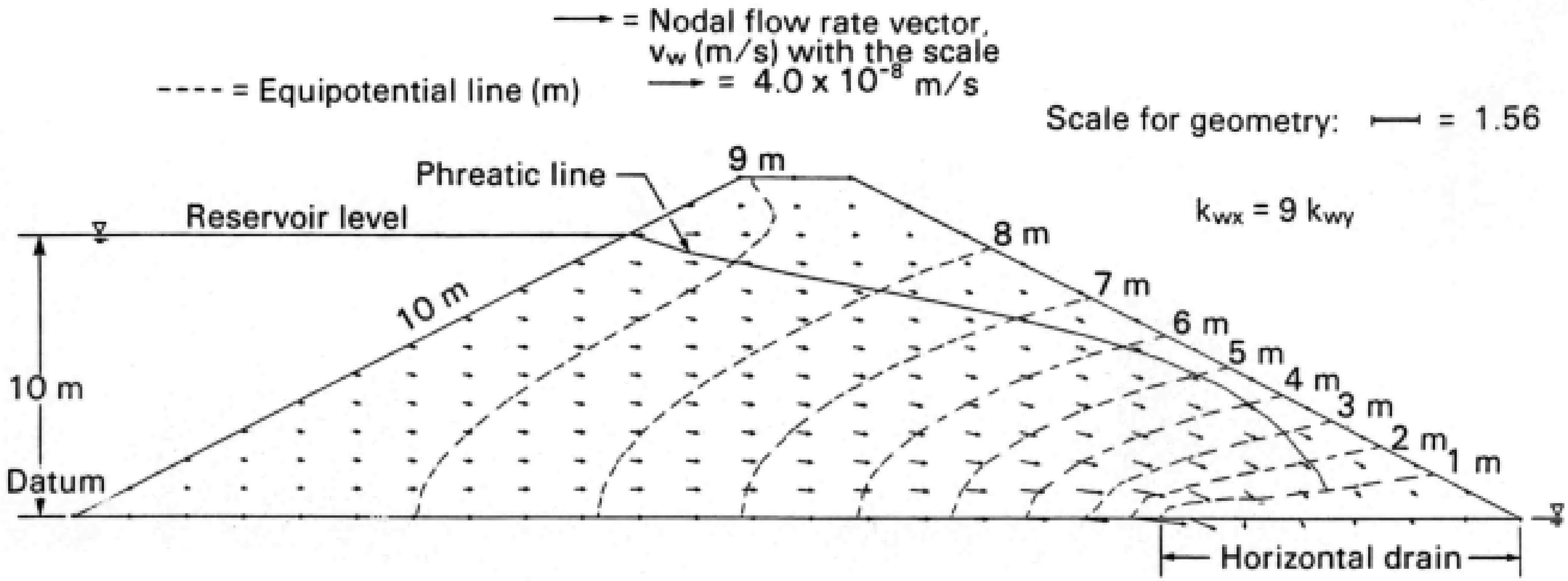


Pressure contours have little physical meaning

(b) Contours of pore-water pressure heads (isobars) through the dam

Isotropic earth dam with a horizontal drain

Compacted soil may have k_h = 9 to 16 times k_v

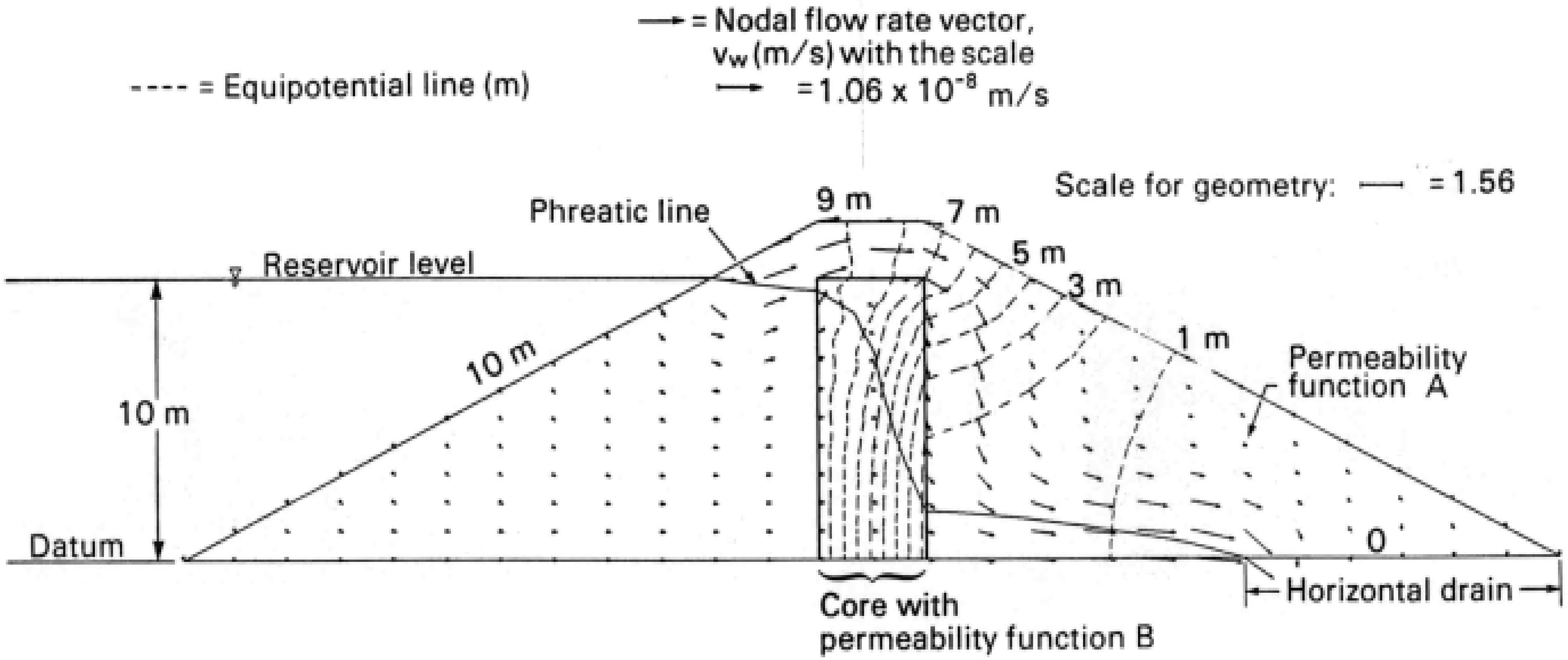


(a) Equipotential lines and nodal flow rate vectors throughout the dam

Anisotropic earth dam with a horizontal drain

Equipotential lines and the zero pressure line (phreatic surface) are of most relevance

UST Unsaturated Soil Technology



(a) Equipotential lines and nodal flow rate vectors throughout the dam

Isotropic earth dam with a core and a horizontal drain

Equipotential lines show energy dissipation is through the core but most of the flow is over the top of the core

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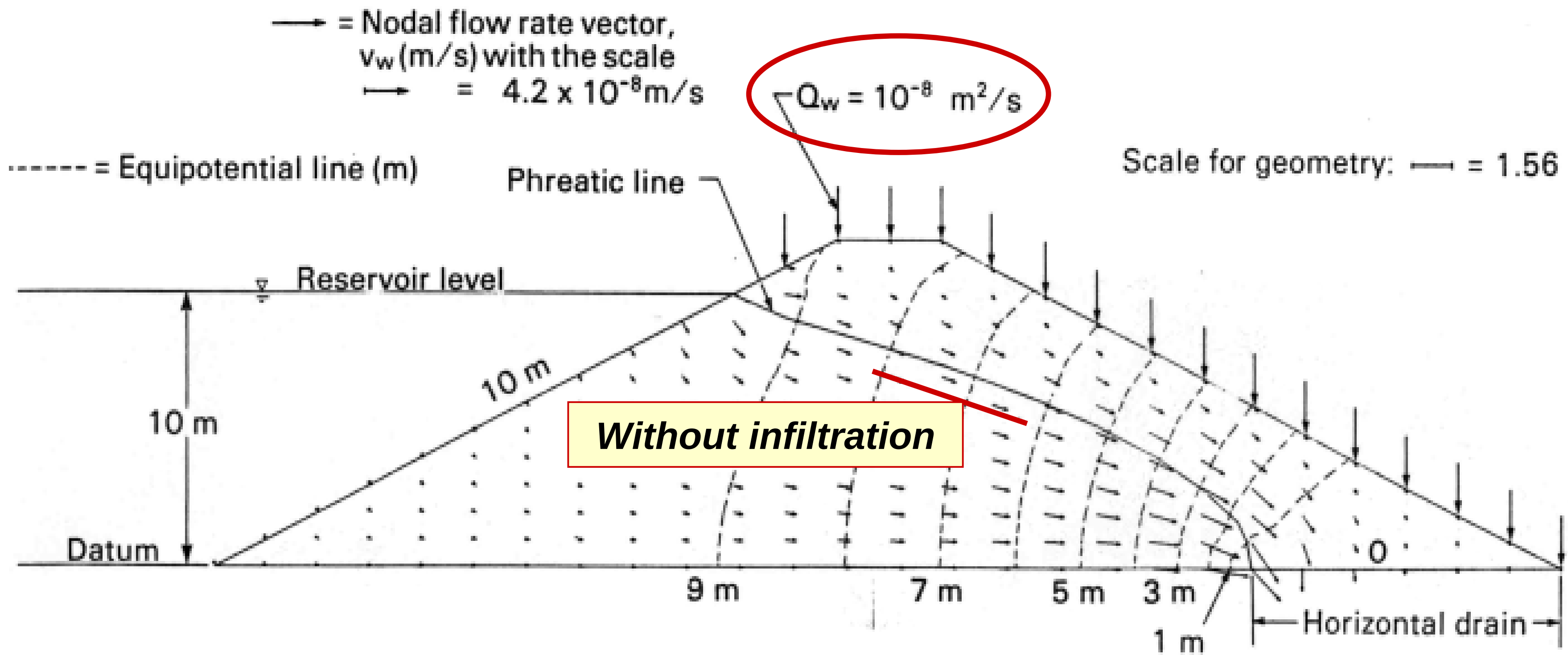

(a) Equipotential lines and nodal flow rate vectors throughout the dam

Isotropic earth dam with a horizontal drain under steady state infiltration

Rainfall causes the phreatic line to rise

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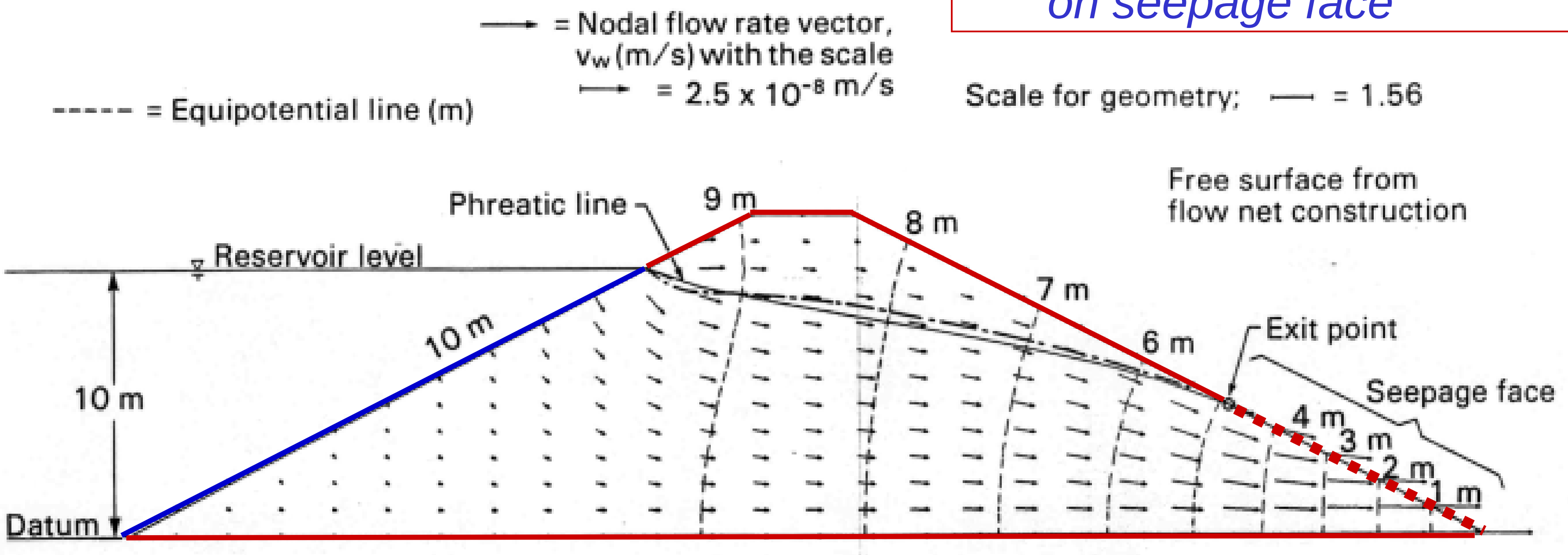
$h = u_w/(\text{unit weight}) + Y$

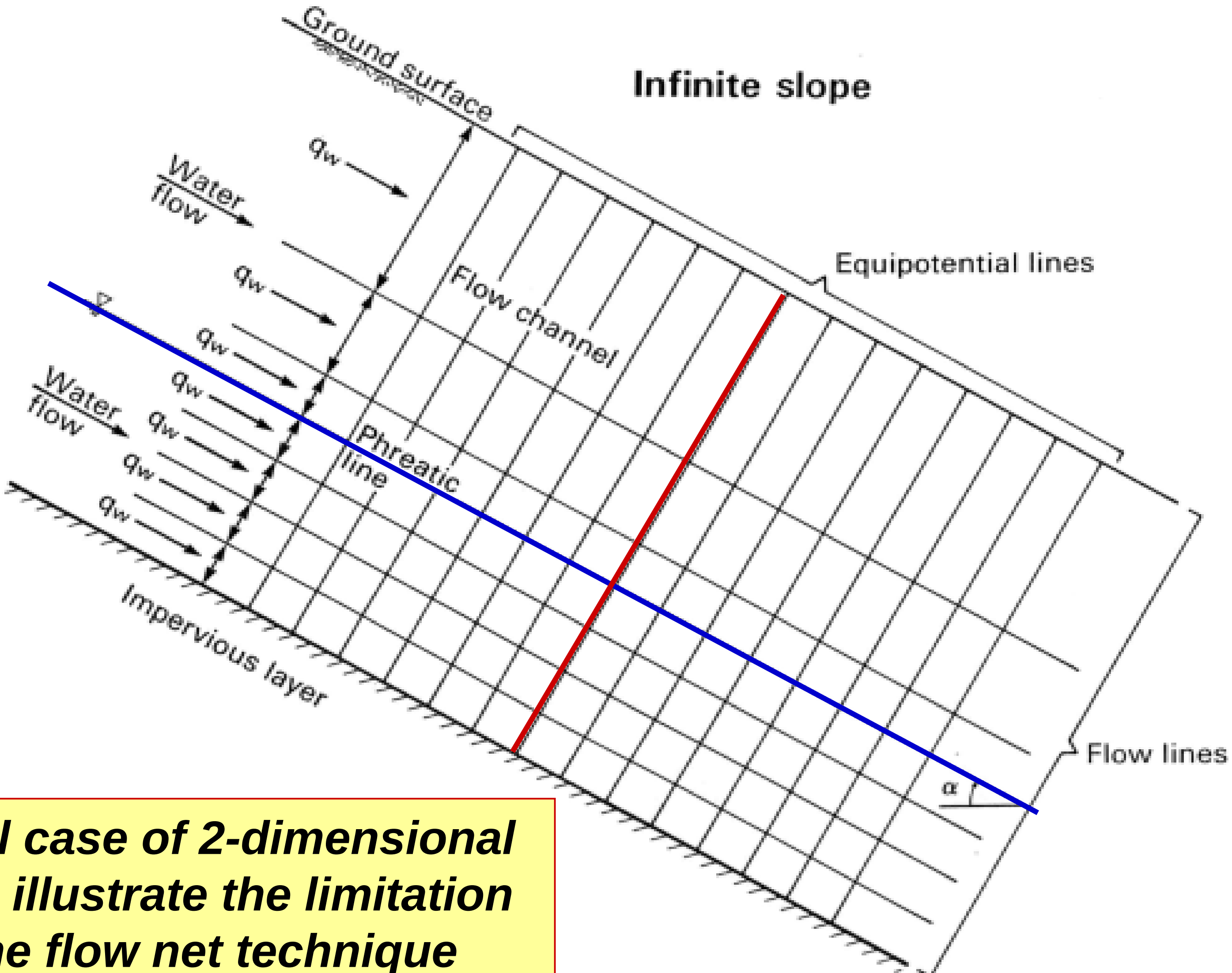
$h = Y$ since $u_w = 0$ on seepage face



(a) Equipotential lines and nodal flow rate vectors throughout the dam

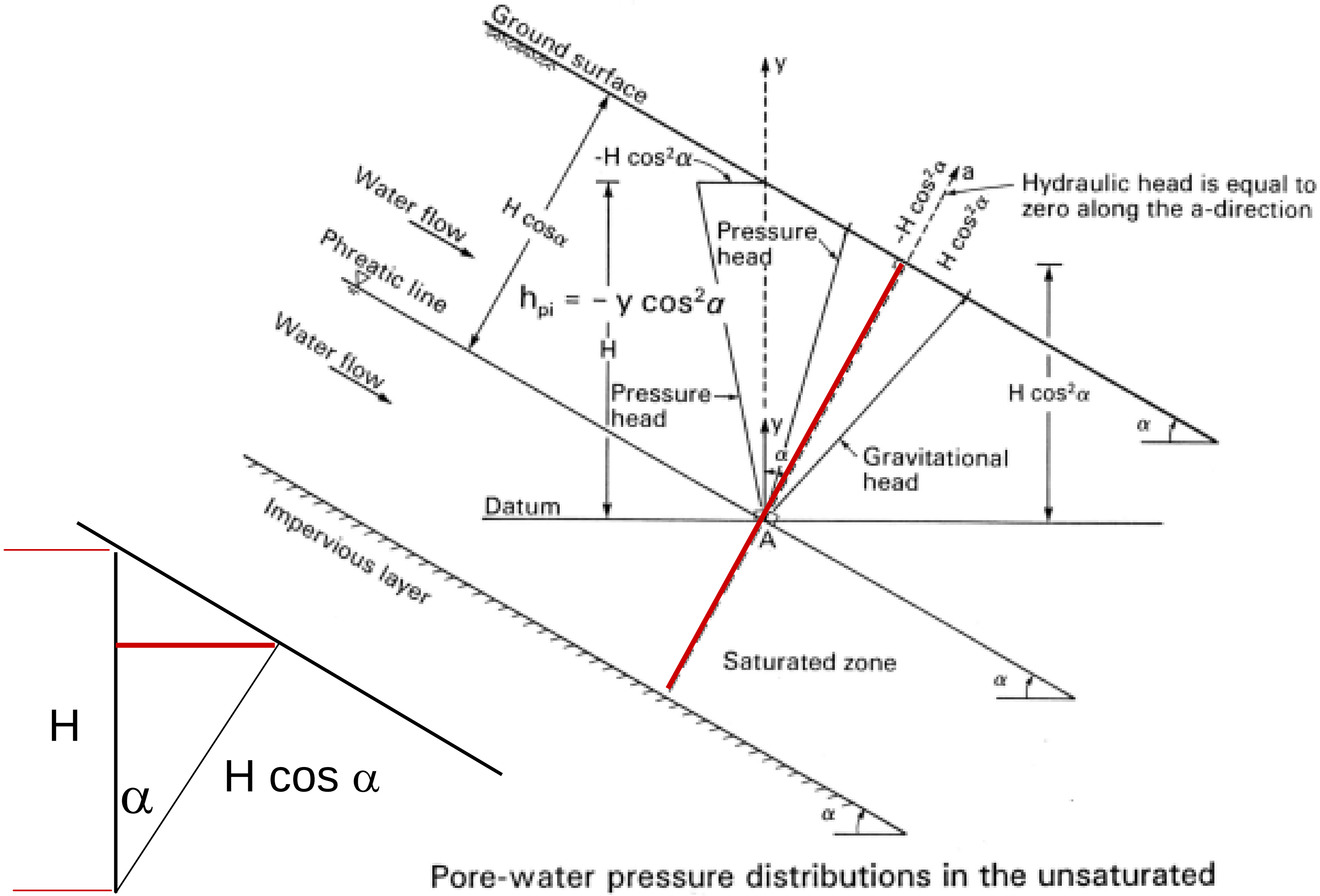
Isotropic earth dam with an impervious lower boundary

Seepage face gives rise to a special type of boundary condition



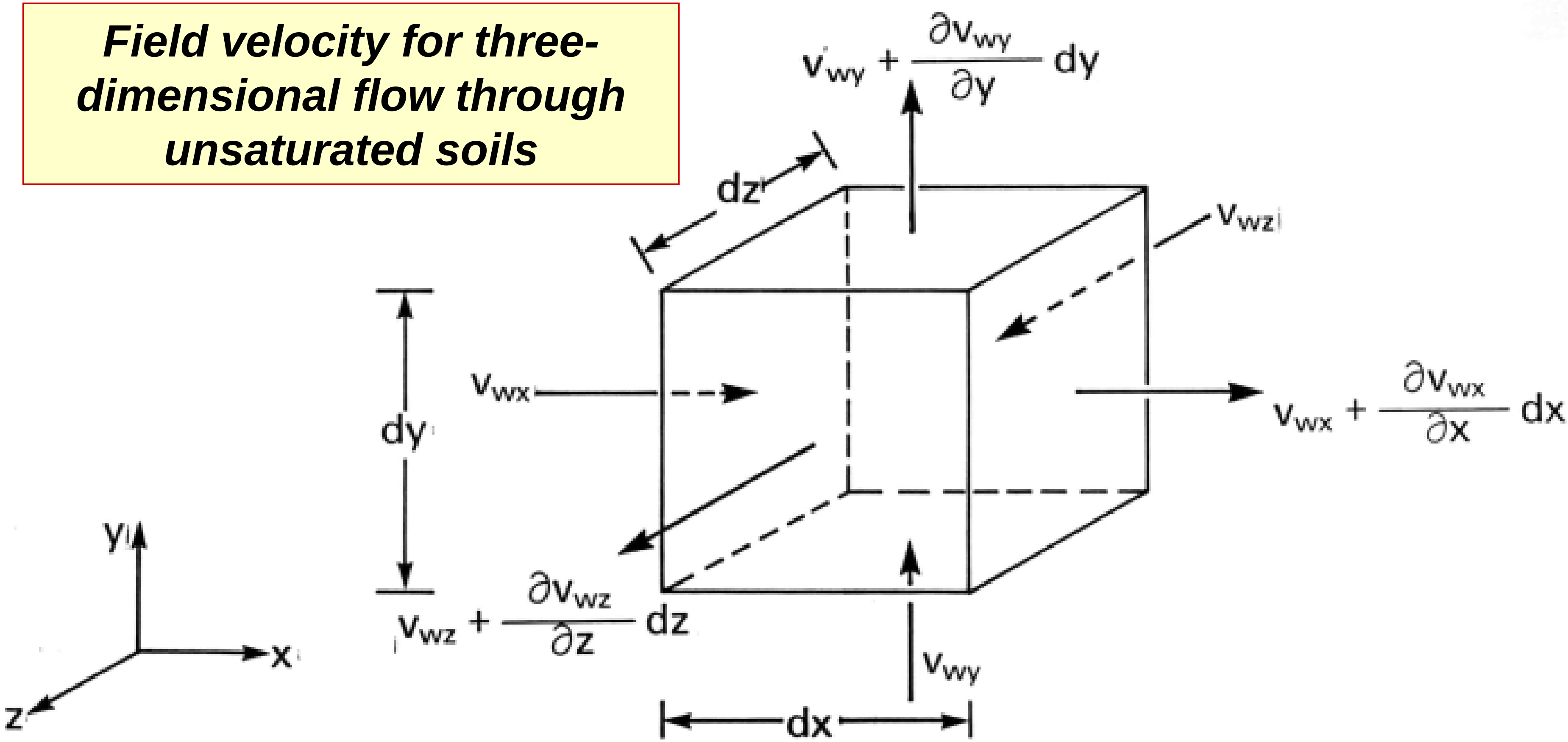
Special case of 2-dimensional flow to illustrate the limitation of the flow net technique

Steady state water flow through an infinite slope



Pore-water pressure distributions in the unsaturated zone of an infinite slope during steady state seepage

Field velocity for three-dimensional flow through unsaturated soils



Three-dimensional steady state water flow through an unsaturated soil element

Derivation of the partial differential equation for three-dimensional flow through unsaturated soils

$$\left(v_{wx} + \frac{\partial v_{wx}}{\partial x}dx - v_{wx}\right) dy\, dz + \left(v_{wy} + \frac{\partial v_{wy}}{\partial y}dy - v_{wy}\right) dx\, dz + \left(v_{wz} + \frac{\partial v_{wz}}{\partial z}dz - v_{wz}\right) dx\, dy = 0$$

where:

v_{wz} = water flow rate across a unit area of the soil in the z- direction

$$\left(\frac{\partial v_{wx}}{\partial x} + \frac{\partial v_{wy}}{\partial y} + \frac{\partial v_{wz}}{\partial z}\right) dx\, dy\, dz = 0$$

$$\frac{\partial}{\partial x}\left\{k_{wx}(u_a - u_w)\frac{\partial h_w}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{k_{wy}(u_a - u_w)\frac{\partial h_w}{\partial y}\right\} + \frac{\partial}{\partial z}\left\{k_{wz}(u_a - u_w)\frac{\partial h_w}{\partial z}\right\} = 0$$

where:

$k_{wz}(u_a - u_w)$ = water coefficient of permeability as a function of matric suction

$\partial h_w / \partial z$ = hydraulic head gradient in the z -direction

$$k_{wx}\frac{\partial^2 h_w}{\partial x^2} + k_{wy}\frac{\partial^2 h_w}{\partial y^2} + k_{wz}\frac{\partial^2 h_w}{\partial z^2} + \frac{\partial k_{wx}}{\partial x}\frac{\partial h_w}{\partial x} + \frac{\partial k_{wy}}{\partial y}\frac{\partial h_w}{\partial y} + \frac{\partial k_{wz}}{\partial z}\frac{\partial h_w}{\partial z} = 0$$