8. Cam Clay Model

8-1. Cam Clay Model: A brief introduction

The Cam Clay Model was developed by researchers of the University of Cambridge (hence its name) in the 1950s-1960s (Roscoe et al., 1958, 1963; Roscoe and Burland, 1968). It is arguably the most influential soil model that has ever been proposed, and there are still many adherents around the world. Many of the more sophisticated models proposed in more recent years adopt the Cam Clay Model as their backbone.

The Cam Clay Model is particularly popular because, put simply, everything seems consistent and beautiful in it. It is also suited to computer coding, and in some simple problems, it also provides an analytical solution. However, it is sometimes caricatured as a 'student model', which is good for students' study but too much idealised to be useful in practice. Be that as it may, we still need a simple framework to interpret and describe soil behaviour, and the Cam Clay Model is useful as a gateway to more advanced interpretations. And, you are in fact students! So let us learn the Cam-Clay Model this week.

Now remind yourself of the ingredients of elastoplastic models that we studied last week:

- Elasticity
- Yield criteria
- Plastic potential
- Hardening rule

These will appear in a similar way as in the Mohr-Coulomb model. Only the last item, a hardening rule, is more complicated than in the Mohr-Coulomb model.
8-2. Cam Clay Model: How it works

What we are going to study here is called the Modified Cam Clay model. We will study later what was modified from the original version.

(i) Isotropic compression and swelling lines (i.e. $q = 0$)

The isotropic compression and swelling lines are the backbone of the model.

This is as we have already know.

Three material parameters are introduced here:

- $\lambda$, $\kappa$: Very similar to compressibility index, $C_c$ and swelling index, $C_s$, but note that they are defined in terms of $\ln p'$, not $\log p'$.

- $N$: The value of specific volume $v$ at $p' = 1$.

So the Isotropic NCL can be expressed as

$$v = N - \lambda \ln p'$$

An internal variable is also introduced.

- $p_0'$: Yield stress (in isotropic compression)
(ii) Yielding: Example of isotropic consolidation and triaxial compression

The (cross-section of) yield surface in the Modified Cam Clay Model is elliptic:

\[ f = q^2 - M^2 [p'(p'_0 - p')] \]

The yield function never becomes positive. It is zero if the stress is on the yield surface. It is negative if the stress is inside the yield surface.

\( M \) is the slope of the Critical State Line (CSL):

\[ q = Mp' \]

The size of the yield surface is determined by \( p'_0 \) and \( M \).

\( M \) is called the Critical State Parameter and it is considered a material constant. The value of \( p'_0 \) is obviously not a constant. It is an internal variable of the model.
(iii) Plastic potential and flow rule

Co-axiality:
In the Cam Clay Model, as in the Mohr-Coulomb model that we studied last week, co-axiality of stress and strain is assumed. That is, the components of $\delta \epsilon^p$ along the $q$ and $p'$-axes are assumed to be $\delta \epsilon^p_q$ and $\delta \epsilon^p_{p'}$, respectively.

Associated flow rule:
Plastic potential is identical to the yield surface and is expressed by the same function.

Plastic potential:
$$g = q^2 - M^2 [p'(p_0' - p')]$$

Plastic potential contours
(iv) Hardening rule

Now we know when we get the plastic strain and in which direction. The question left then is, how much strain? In other words, we now know the direction of the plastic strain increment vector, but we do not know its length.

In fact, based on what we have looked at already, you can in fact derive the answer to the above question. We do not need to add extra assumptions.

Imagine the example shown in the diagram.

For a given stress increment, we now know the destination in the $v - p'$ space. It means, we know $\delta v$.

The change in the specific volume, $\delta v$ is, in essence, the volumetric strain.

$$
\delta v^{p} = -\frac{\delta v^{\nu}}{v}
$$

With $\delta v^{p}$ known now, $\delta v^{\nu}$ is also known from the flow rule.

Looking at the bottom figure, the expansion of the yield surface (i.e. change of $p'_{0}$) is linked to the plastic volumetric change. Such a rule governing the evolution of the yield surface is called a hardening rule.

The Cam Clay Model is an isotropic-hardening model, in which the yield surface only expands/shrinks without translation or rotation.
(v) The Critical State

Now let us consider the meaning of the Critical State Line (CSL).

Consider a path such as indicated in the diagram. When the yield surface is still small, the plastic strain increment is not parallel to the $q$-axis, meaning that the plastic volumetric strain increment is not zero; the volume is changing.

However, as the stress reach the Critical State, the plastic strain vector is parallel to the $q$-axis. There is no further volumetric change. Hence, no hardening of the yield surface. Because the stress point cannot exceed the yield surface, the path is ‘stuck’ at this point, unless unloading occurs.

This is an ultimate state, from which no change in volume and stress is possible.

Such a Critical State is believed to be seen for granular soils and some low plasticity clays at large shear strains. However, there are also a variety of soils in which a clear Critical State is difficult to observe.

If we draw the Critical State Line (CSL) in the $v – \ln p'$ space, it is parallel to the Normal Compression Line (NCL).
In this diagram, the yield surface (the arcs) evolving according to the volume change $\nu$ forms a surface outside which a state can never be. This surface is called the State Boundary Surface (SBS).

An SBS defines strength. Probably you know that soil in general has higher strength for denser (smaller $\nu$) states and lower strength for looser states (larger $\nu$). The above 3-D representation takes into account these characteristics in a single snapshot.

In this 3-D representation, trajectories of the states in conventional soil tests are easily understood. Let us have a look at the following examples.

- Consolidation tests: Isotropic and $K_0$
- Triaxial compression tests: Drained and undrained
Example 1: Consolidation

- **q-p’ projection**
- **Isotropic consolidation**

Example 2: Triaxial compression

**Drained triaxial compression:**
The \( q - p' \) path is controlled, and change in \( v \) is observed.

**Undrained triaxial compression:**
The \( v - q' \) path is controlled, and change in \( p' \) is observed.
(vii) Elasticity within yield surface

In the Cam Clay Model, isotropic elasticity is typically assumed. This is the same as the Mohr-Coulomb model that we studied last week.

\[
\begin{pmatrix}
\frac{d\varepsilon_p}{dp'} \\
\frac{d\varepsilon_q}{dp'}
\end{pmatrix} =
\begin{bmatrix}
1/K & 0 \\
0 & 1/3G
\end{bmatrix}
\begin{pmatrix}
dp' \\
dq
\end{pmatrix}
\]

In the previous illustration, the elastic volume change in the Cam Clay Model was expressed by the parameter \( \kappa \). Note the relationship between \( K \) and \( \kappa \). Constant \( \kappa \) means non-constant \( K \), as \( v \) and \( p' \) is not constant.

So the Cam Clay Model adopts **nonlinear elasticity**, at least for unloading – reloading. Because the shear modulus \( G \) is independent from \( B \), its value can be independently set. It is important to note, however, that the Poisson’s ratio, \( \nu \), could become negative at small \( p' \) if \( G \) is kept constant; see the following relationship.

\[
\nu = \frac{3K - 2G}{2(3K + G)}
\]
Summary of the Cam Clay Model

For all its sophistication, only five model parameters are required.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>Elastic modulus (volumetric)</td>
</tr>
<tr>
<td>$G$</td>
<td>Elastic modulus (shear)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>A parameter to determine NCL</td>
</tr>
<tr>
<td>$N$</td>
<td>A parameter to determine NCL</td>
</tr>
<tr>
<td>$M$</td>
<td>The Critical State Parameter</td>
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</tbody>
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The equations for the yield surface, plastic potential (both elliptic) and the NCL are all explicitly given. The stress and strain developments can be calculated quantitatively in an incremental manner by the computer.

In this lecture, the mathematical exercise to formulate the incremental stiffness matrix is omitted. For those interested, there are some books (e.g. Wood, 1990; Potts and Zdravkovic, 1999).

Finally, what is the difference between the Modified Cam Clay Model and the original? It is the difference in the shape of the yield surfaces.

In the original Cam Clay Model, the yield surface has a logarithmic spiral shape. This shape was theoretically derived based on an energy equation. However, it is an inconvenient shape, as it has a pointy tip: i.e., it predicts an unrealistic plastic shear strain increments against isotropic compression. Today, the Modified Cam-Clay Model is predominantly used as it is or as a base for more sophisticated model.
References