Analysis and Optimization for Vibration of Laminated Composite Rectangular Plates with General Elastic Support Conditions

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Introduction

The light-weight advanced composites are targeted toward structural applications in aerospace, marine and automobile areas for the requirements of higher performance and lower operational cost. The composites are also known as preferred materials because of their tailoring capability for bringing the optimization into the design process. In practical applications, structural components such as plates and shells quite often have non-idealized boundary conditions which act as additional restraints from vibration point of view.

Many technical papers have resulted [1] on the vibration problem of plates with idealized classical boundary conditions. In practical situation, the edge restraints are not perfectly compatible with mathematically defined boundary conditions, and would rather be elastic to some extent. The past literature therefore includes some studies on vibration of laminated composite plates having elastic edge springs [2-5] but to the authors’ knowledge, there have been no papers to consider the lay-up optimization of such plates.

In the present study, two of optimization methods are applied to symmetrically laminated rectangular plates having elastic edge restraints on the boundary.

The comparative results will be drawn from the analysis of the methods’ calculation performances. The first method, layerwise optimization (LO) approach [6-8] provides a simple design procedure for composite structural optimization which may be used to maximize the fundamental frequencies of the plates. Another method applied in this study, Genetic Algorithm [11-12], is used to approach the optimum angle arrangement avoiding convergence to local optima in the optimization process. The design variables are taken for a set of fiber orientation angles in the symmetric layers.

Analysis of laminated plates with elastic edge springs

Figure 1 shows a laminated composite rectangular plate of dimensions $a \times b$ with thickness $h$, having elastic line restraints along the edges. Each edge is elastically restrained by translational spring and/or rotational spring. A Cartesian coordinate system $x, y$ is taken with its origin at the center of the plate. The direction of the fibers and the transverse direction to the fibers are denoted by $L$ and $T$, respectively, and the fiber orientation angle between the $x$ and $L$ axes is denoted by $\theta$. Each layer is considered to be macroscopically orthotropic.

The strain energy due to bending deformation is given by

$$U = U_1 + U_2$$

where $U_1$ and $U_2$ are the strain energy stored in a plate and the energy in the elastic restraints, respectively. The $U_1$ is given by

$$U_1 = \frac{1}{2} \int_A \{\kappa\}^T [D] \{\kappa\} dA$$

where

$$\{\kappa\} = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} [D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$$

are the curvature vector and the stiffness matrix, respectively. The $w$ is the deflection, and the element in the stiffness matrix is

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{D}_{ij}^{k} \left( z_k - z_{k-1} \right)$$

with $N$ being the number of layers.
The elastic constants $\mathcal{G}_{ij}^{k}$ in eq. (4) are determined by a fiber orientation angle and

$$Q_{ijkl} = \frac{E_k}{V_k V_l}, \quad Q_{ij} = \frac{E_k}{V_k V_l},$$

$$Q_{oo} = \frac{E_k}{V_k V_l}, \quad Q_{oo} = \frac{E_k}{V_k V_l}$$

in the $k$th layer [9].

The kinetic energy $T$ of the plate is given by

$$T = \frac{1}{2} \int_{\Omega} \rho \left( \frac{\partial w}{\partial t} \right)^2 dV$$

where $\rho$ is an average mass per unit volume [kg/m$^3$].

The energy stored in the elastic restraints is written by

$$U = \frac{1}{2} \int_{\Omega} \left[ \int_{-\frac{h}{2}}^{\frac{h}{2}} k_{ij} \left( \frac{\partial w}{\partial x} \right)^2 + \int_{-\frac{h}{2}}^{\frac{h}{2}} k_{ij} \left( \frac{\partial w}{\partial y} \right)^2 \right] dx dy + \int_{\Omega} \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) dx dy$$

(7)

where $k_i (i=1,2,3,4)$ are stiffness of translational springs [N/m$^2$] and $k_{ij} (i=1,2,3,4)$ are of rotation springs [N] per unit length.

The deflection is assumed in sinusoidal time variation as

$$w(x,y,t) = \pi(x,y) \sin \alpha t = \frac{h}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} X_m(x) Y_n(y) \sin \alpha t$$

(8)

In the following analysis, non-dimensional quantities are used for simplicity of analysis.

$$\xi = 2x/a, \quad \eta = 2y/b$$

$$\Omega = \omega \sqrt{\frac{\rho}{E(h^3 / 12(1-v^2))}} \frac{h}{b}$$

(9)

$$\bar{E} = \frac{k}{D_0} \bar{E}_n + \frac{k_{ij}}{D_0} \alpha = a/b$$

After rewriting equations (1)-(8) by using eqs. (9), the functional is minimized with respect to unknown coefficients $C_{mn}$

$$\frac{\partial}{\partial C_{mn}} (U_{mn1} + U_{mn2} - T_{max}) = 0$$

(10)

and the frequency equation is obtained as

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \frac{D_0}{D_{ff}} \bar{E} \left( i \omega \mu \right)^2 + \frac{D_0}{D_{ff}} \bar{E} \left( i \omega \mu \right)^2 + \frac{D_0}{D_{ff}} \bar{E} \left( i \omega \mu \right)^2 \right] + \frac{1}{2} \left[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \theta_{mn}}{\partial x} \right) \left( \frac{\partial \theta_{mn}}{\partial x} \right) dx + \frac{1}{2} \left[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left( \frac{\partial \theta_{mn}}{\partial y} \right) \left( \frac{\partial \theta_{mn}}{\partial y} \right) dx \right] \right]$$

(11)

$$- \frac{1}{16} \left[ \int_{\Omega} \frac{\partial \theta_{mn}}{\partial x} \left( \frac{\partial \theta_{mn}}{\partial x} \right) dx \right] = 0$$

where $\Gamma$ is the definite integral that can be evaluated exactly by

$$I_{mn1}^{(min)} = \phi_{mn1}^{(min)} \times \phi_{mn2}^{(min)}$$

(12)

that is a product of two integrals in $x$ and $y$ directions, such as

$$\phi_{mn}^{(min)} = \int_{\Omega} \left( \frac{\partial \theta_{mn}}{\partial x} \right) \left( \frac{\partial \theta_{mn}}{\partial x} \right) dx$$

The integrals along the elastic edges are evaluated by

$$\psi_{mn}^{(e)} = \int_{\Omega} \beta_{mn}^{(e)} \left( \frac{\partial \theta_{mn}}{\partial x} \right) dx$$

(14)

In eqs. (13)-(14), the $X_{ma}(\xi)$ and $Y_{ma}(\eta)$ are the functions that satisfy the kinematical boundary conditions at the edges, and are expressed in our previous studies [10] for classical boundary conditions by

$$X_{ma}(\xi) = \cos \left( \xi \right) \cos \left( \xi \right)$$

(15)

$$Y_{ma}(\eta) = \sin \left( \eta \right) \sin \left( \eta \right)$$

that BC1, BC2, BC3, and BC4 are the boundary index [10]. The classical boundary conditions may be one of clamped (denoted by C), simply supported (S) and free (F) edges along the boundary, and the total edge condition of a rectangular plate was given by a combination of C, S, and F. In addition to such classical boundary conditions, translational and rotational elastic restraints can be added to each of the four edges, allowing the model to be capable of calculating the frequency of laminated plate with non-ideal boundary condition.

**Methods of optimization**

This section briefly introduces an optimization method used in the study. The design variables are taken to be a set of fiber orientation angles in the layers of the upper (and lower) half of the cross-section:

$$\theta_1 / \theta_2 / ... / \theta_1 / ... / \theta_N$$

(16)

where $\theta_k$ is a fiber orientation angle in the $k^{th}$ layer $(k=1$: outermost, $k=N$: innermost) and the subscript “s” denotes symmetric laminating. The optimization problem may be written in standard form as

$$\text{Find} : \bar{\theta} = \left[ \theta_1 / \theta_2 / ... / \theta_1 / ... / \theta_N \right]_s$$

(17)

which maximize: $\Omega = \omega \left( \bar{\theta} \right)$ (fundamental frequency) subject to the constraints: $-90^\circ \leq \theta_k < 90^\circ (k=1,2,...,N)$

**Layerwise Optimization**

In the layerwise optimization (LO) approach, use of a simple physical observation: In the bending of laminated plates, the outer layer has a greater stiffening effect than the inner layer and therefore has a greater influence on the natural frequency. Thus, the optimum stacking sequence $\left[ \theta_1 / \theta_2 / ... / \theta_1 / ... / \theta_N \right]_s$ can be obtained by determining the optimum fiber angle for each layer sequentially in the order from the outermost to the innermost layer. It is expected that several iterative cycles may be necessary in the LO approach to reach to the converged optimum solution [6]-[8].

**Genetic Algorithm**

In Genetic Algorithm, the concept of natural selection of
living organisms (survival of the fittest) is applied to the optimizing processes. By using probabilistic rules, GA became a great optimization tool in dealing with continuous and discrete valued design variable by using only desired objective function. Unlike gradient based algorithm which may converge to locally optimal regions of design space, GA is less likely to get trapped in local optimal areas because it works with a number of populations instead of a single point search.

RESULTS AND DISCUSSIONS

Since this study deals with a wide range of boundary restraints continuously from a totally free plate to a totally clamped plate, the following notation is used;

F: Free edge (no restraints)
FE: Deflection w is elastically restrained by translational stiffness and rotation is not restrained.
S: Simply supported edge (deflection is zero)
SE: Deflection w is set to zero, and rotation is elastically restrained by rotational stiffness.
C: Clamped edge (deflection and rotation are zero)

These notations are used to define the plate boundary condition starting from left hand edge to upper edge in counter-clock-wise direction.

Two sets of four examples are presented in the following results. The first set (Ex.1-4) is used to compare the natural frequencies of square cantilevered plates and (Ex.5-8) is used to evaluate the effect of various degrees of elastic restraints along four edges of square plates. The examples are as follow:

<table>
<thead>
<tr>
<th>Ex.1</th>
<th>SE($\tau_{1} - 1$)-F-F-F</th>
<th>Ex.2</th>
<th>SE($\tau_{1} - 10^6$)-F-F-F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex.3</td>
<td>SE($\tau_{1} = 10^6$)-F-F-F</td>
<td>Ex.4</td>
<td>C-F-F-F</td>
</tr>
<tr>
<td>Ex.5</td>
<td>FE($\tau_{1} = 10^6$)-FE($\tau_{1} = 10^6$)-FE($\tau_{1} = 10^6$)</td>
<td></td>
<td></td>
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<tr>
<td>Ex.6</td>
<td>SE($\tau_{1} = 10^6$)-S-FE($\tau_{1} = 10^6$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ex.7</td>
<td>C-FE($\tau_{1} = 10^6$)-FE($\tau_{1} = 10^6$)-F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ex.8</td>
<td>C-SE($\tau_{2} = 10^6$)-SE($\tau_{3} = 10^6$)-FE($\tau_{4} = 10^6$)</td>
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<td></td>
</tr>
</tbody>
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Table 1. Convergence rate verification for Ex.3, 5, 8

<table>
<thead>
<tr>
<th>No. of Terms</th>
<th>$\Omega_i$</th>
<th>$\Omega_i$</th>
<th>$\Omega_i$</th>
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<tbody>
<tr>
<td>6x6</td>
<td>10.19</td>
<td>24.06</td>
<td>51.3</td>
<td>65.79</td>
</tr>
<tr>
<td>8x8</td>
<td>10.15</td>
<td>24.01</td>
<td>50.67</td>
<td>65.51</td>
</tr>
<tr>
<td>10x10</td>
<td>10.14</td>
<td>24</td>
<td>50.65</td>
<td>65.47</td>
</tr>
<tr>
<td>12x12</td>
<td>10.14</td>
<td>24</td>
<td>50.65</td>
<td>65.47</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No. of Terms</th>
<th>$\Omega_i$</th>
<th>$\Omega_i$</th>
<th>$\Omega_i$</th>
<th>$\Omega_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex.5</td>
<td>FE($k=10^6$)-FE($k=10^6$)-FE($k=10^6$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. of Terms</td>
<td>$\Omega_i$</td>
<td>$\Omega_i$</td>
<td>$\Omega_i$</td>
<td>$\Omega_i$</td>
</tr>
<tr>
<td>6x6</td>
<td>6.743</td>
<td>23.26</td>
<td>43.4</td>
<td>53.52</td>
</tr>
<tr>
<td>8x8</td>
<td>6.737</td>
<td>23.25</td>
<td>43.25</td>
<td>52.89</td>
</tr>
<tr>
<td>10x10</td>
<td>6.736</td>
<td>23.25</td>
<td>43.23</td>
<td>52.88</td>
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<tr>
<td>12x12</td>
<td>6.736</td>
<td>23.25</td>
<td>43.23</td>
<td>52.88</td>
</tr>
</tbody>
</table>

| Ex.8         | C-SE($k=10^6$)-SE($k=10^6$)-FE($k=10^6$) |
| No. of Terms | $\Omega_i$ | $\Omega_i$ | $\Omega_i$ | $\Omega_i$ |
| 6x6          | 65.42      | 96.63      | 155.4      | 173.7      |
| 8x8          | 65.35      | 96.53      | 153.2      | 173.3      |
| 10x10        | 65.33      | 96.51      | 153.2      | 173.3      |
| 12x12        | 65.33      | 96.5       | 153.1      | 173.3      |

Convergence characteristics of the solution (8) is tested in Ex.3, 5 and 8 (a/b=1, [30/30/30]s) for the number of terms $M=N=6$, 8, 10 and 12 in Table 1. As the number of terms is increased, the solutions decrease monotonically from above, and almost converge within four significant figures as in Table 1. Based on the result, the term $M=N=10$ is adapted in the following results.

Figure 2 shows variations of the lowest five frequency parameters of symmetrically laminated square plates from a totally free case to a totally clamped case. The stiffness of torsional spring is uniform for all four boundaries and is varied of from $\tau_{1} = 1$ to $10^6$ and $\tau_{1} = 1$ to $10^6$. When $\tau_{1} = 10^6$ and $\tau_{1} = 10^6$ the frequency parameters coincide with SSSS and CCCC plate respectively.

Comparisons are made to illustrate that the square plates with the present optimum lay-ups actually give higher fundamental frequencies than plates with other lay-ups for Ex.1~Ex.8. In Fig.3 (LO-optimized) and Fig.4 (GA-optimized), it is observed that all of the present optimum solutions have higher frequencies, without exception, than the plates with the five typical lay-ups. This shows the effectiveness of the optimization techniques used in this study.

Optimized fundamental frequency and time difference between LO and GA are also computed. In Table 2, percentage differences of optimized frequency obtained from GA and LO are written. It can be seen that despite the trivial percentage difference in some example, the overall results shows that both LO and GA methods have almost equivalent frequency optimization effect.

![Fig.2 Variation of frequency of laminated plate with elastic edge restraints (a/b=1, [30/30/30]s)](image-url)
An optimization procedure was applied to the vibration design problem of laminated rectangular plates with elastic edge restraints. A layerwise optimization (LO) and Genetic Algorithm (GA) were applied to the prediction of the optimum lay-ups in order to find the maximum natural frequency in the lowest mode. In the analysis, a Ritz method was used to calculate the frequency parameters in short computation time. Eight different sets of elastic edge restraints have been considered in numerical examples. The results demonstrated excellent capability of LO & GA approach. The comparisons between two method show that both have almost equivalent optimization effect with the difference that LO consumes shorter calculation time than GA in the optimization process.

**REFERENCES**


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**Fig.3** Frequency comparison between LO-Optimized lay-up and other 5 typical lay-ups

**Fig.4** Frequency comparison between GA-Optimized lay-up and other 5 typical lay-ups

**Table2.** % difference of Opt. frequency (GA>LO) (%)

<table>
<thead>
<tr>
<th>Example</th>
<th>GA</th>
<th>LO</th>
<th>GA/LO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ex2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ex3</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Ex4</td>
<td>0</td>
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<tr>
<td>Ex5</td>
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<tr>
<td>Ex6</td>
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<td>0</td>
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<tr>
<td>Ex7</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ex8</td>
<td>0</td>
<td>0</td>
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</tbody>
</table>

However, as the time used in each method is compared in table3, it can be seen obviously that GA (with 400 populations, 400 generations) consume considerably much longer time than LO in completing the optimization. Thus, it can be said that LO gives relatively good frequency optimization effect with shorter computational time comparing to GA method.

**CONCLUSIONS**

An optimization procedure was applied to the vibration analysis, a Ritz method was used to calculate the frequency parameters in short computation time. Eight different sets of elastic edge restraints have been considered in numerical examples. The results demonstrated excellent capability of LO & GA approach. The comparisons between two method show that both have almost equivalent optimization effect with the difference that LO consumes shorter calculation time than GA in the optimization process.