**Introduction**

It is well-known that simple Euler buckling modes can occur if any axial stresses are applied on the cross section of a slender beam without any restrictions surrounding the beam. Moreover in the case of beams on an elastic foundation, higher-order buckling modes may arise1,2).

The purpose of this study is to analytically investigate the helical buckling characteristics of a rod embedded in an elastic medium under the various conditions in terms of structural mechanics.

**Analytical Model**

As denoted in Fig.1, it would be considered that a buckled rod which has the circular cross-section lies in the centre of the elastic medium. An external tensile or compressive force $T$ and external end moments $M$ are acting on the cross-section of a rod. Assume that the helical buckling would occur under two different restraints: model A is encircling coil spring model and model B is a simplified DNA-like model.

On the occasion of computing the total potential energy of the model, A Cosserat rod’s energy formulation3) is used here in order that the problem can be cast spatially in terms of three degrees of freedom represented by the internal torsion, $\tau_s$, the helical angle, $\beta$, and the helical radius, $\rho$. Three equilibrium equations, in terms of the total potential energy $V$, are obtained as $\partial V / \partial \tau_s = 0$, $\partial V / \partial \beta = 0$ and $\partial V / \partial \rho = 0$.

**Energy Formulae**

According to the Fig.1, the parameters to determine the helical shape can be derived from the principal of the least energy. The total energy of the analytical model can be expressed as the sum of the potential and strain energies as follows: (1) The potential energy of tensile or compressive force $V_T$, (2) The potential energy of the end moment $V_M$, (3) The bending strain energy $V_B$, (4) The torsional strain energy $V_C$, and (5) The strain energy stored in an elastic medium $V_{G2}$.

Here consider, at first, the potential energy of axial force $T$ and end moment $M$, $V_T$ and $V_M$ can be described as

$$V_T = T \frac{D}{S} = T(1 - \cos \beta) \quad (1)$$

$$V_M = -M \frac{R}{S} = -M \left( \tau_s - \frac{\sin \beta}{\rho} \right) \quad (2)$$

where $R$ is the total angle of radian. That is, Eq. (2) includes the effect of torsion by the end moment.

Whereas, the strain energies are expressed as follows.

$$V_B = \frac{1}{2S} BS \beta^2 = \frac{1}{2} B \sin^2 \beta \quad (3)$$

$$V_C = \frac{1}{2S} CS \tau_s^2 = \frac{1}{2} C \left( \tau_s + \sin \beta \cos \beta \right)^2 \quad (4)$$

where $B = EI$ is the bending stiffness, $E$ is Young’s modulus of a rod, $I$ is the area moment of inertia, and $C$ is the torsional stiffness. And then it is considered the strain energy stored in an elastic medium owing to the helical buckling. The position of a rod is deformed to $\rho$, the radius of the helix. As to the model A, assuming the deformation is completely elastic and cross-sectional, as illustrated in Fig.1, the encircling coil spring can represent the elastic restraints. Hence the stored strain energy $V_{G1}$ can be described as

$$V_{G1} = k \left[ \pi \left( \rho^2 + 2a_0^2 \right) - 4a_0 \left( \rho + a_0 \right) E \left( \frac{2\sqrt{a_0}}{\rho + a_0} \right) \right] \quad (5)$$

where $E(\psi)$ is the elliptic integral of the second kind and $k$ is the spring constants, which are written.

$$E(\psi) = \frac{\pi}{\int} \sqrt{1 - \varphi^2 \sin^2 \theta} d\theta, \quad k = \frac{1}{2} \left( \frac{2a + d}{2a - d} \right) E_i \quad (6)$$

Here $E_i$ is the Young’s modulus of an elastic medium and $a_i = a - d/2$. Regarding model B, the strain energy in restraint part is analogue to simplified spring model.

$$V_{G2} = \frac{1}{2} k (\rho - a_0)^2 \quad k = \frac{d}{b_n} E_i \quad (7)$$

where the spring constant $k$ is mentioned above in this case.

Therefore, the helical buckling from the following equations by means of variation approach can be determined.

$$V_{total} = V_T + V_M + V_B + V_C + V_{G1} \quad (i = 1, 2) \quad (8)$$

$$\frac{\partial V_{total}}{\partial \rho} = 0, \frac{\partial V_{total}}{\partial \beta} = 0, \frac{\partial V_{total}}{\partial \tau_s} = 0 \quad (9)$$

Eq. (9) is the sum of all energies in the analytical model.
Additionally, it would be substituted following non-dimensional parameters into the equation below which is derived from Eq. (9) and Eq. (10).

\[ \rho_N = \frac{\rho}{a}, \quad D_N = \frac{d}{a}, \quad M_N = \frac{M}{B}, \quad T_N = \frac{T}{B}, \quad a = \frac{L}{10} \]  

(10)

where \( a \) is the ratio of Young’s modulus, which indicates that the elastic restraint becomes strong in accordance with \( a \) is close to 1. \( D_N \) is the ratio of cross-sectional area of a rod and outermost layer. The arc-length along with the curve is written

\[ S_N = L \cos \beta, \quad L = \frac{2 \pi \rho}{\sin \beta} \]  

(11)

where \( S_N \) is the pitch of the helix and \( L \) is the arc-length.

**Analytical Results**

The ratio of Young’s modulus \( \alpha \) is varied from 1/1000 to 1, and non-dimensionalized parameters are assigned with regard to external forces \( M_N, T_N \) in order to measure the rate of change of the helical angle \( \beta \) and the helical radius \( \rho_N \). Note that \( D_N \) is fixed at 1/10 and consider the positive area as to \( \beta \). Fig.2 denotes that the areas of existence of a value of \( \beta \) correspond to the combination of external forces \( M_N \), and \( T_N \), respectively. The result is satisfied with the equation \( \partial V/\partial \beta = 0 \). Therefore the result is not depending on an elastic medium. The result indicates that the helical angle exists corresponding to \( M_N \) even if \( T_N \) is tensile (minus value).

In Fig.3, the pitch decreases in response to the increase of the helical radius and of \( \alpha \), which signifies that the harder an elastic restraint becomes, the more gradual the pitch decreases. Compared with Fig.3 (model A) and Fig.4 (model B), how the helical radius, \( \rho \) increases depends on the rate of Young’s modulus \( \alpha \). Note that here 0.95 is the initial length of the spring in the model as denoted in Fig.1. With regard to Fig.5 and Fig.6, which discern the helical radius increases corresponding to the increase of the compressive force, \( T_N \). As well as mentioned above, the helical pitch increases in response to the increment of \( T_N \).

**Conclusion Remarks**

The specific feature of making helical buckling exerted by an external load’s combination (\( T, M \)) is investigated. The elastic binding effect certainly restricts the increase of helical radius and decrease of helical pitch in both cases, however, an elastic restraint’s effect would act on the forming of helical buckling differently, as illustrated in this paper.

**Reference**

1) Brush, D. O. and Almroth, B. O. (1975), Buckling of Bars, Plates, and Shells, McGraw-Hill, Inc., Columbus Ohio, USA.


