1. Introduction

The two-dimensional Schrödinger equation is solved theoretically [1-3] and numerically [4-9] for a wavefunction $\psi$ at position $r$ and time $t$,

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{ \frac{1}{2m} \left(-i\hbar \nabla - qA\right)^2 + qV \right\} \psi,$$

where $V$ and $A$ stand for the scalar and vector potentials, $m$ and $q$ the mass and electric charge of the particle under consideration, $i = \sqrt{-1}$ the imaginary unit. In the presence of a uniform magnetic field, with a Landau gauge, of $A_y = -By$, $A_x = 0$, $A_z = 0$, and by solving the Heisenberg equation of motion,

$$i\hbar \frac{d\hat{X}(t)}{dt} = \left[ \hat{X}(t), \hat{H} \right],$$

where the square bracket $\left[ \cdot, \cdot \right]$ is the commutator, $\hat{H}$ is the Hamiltonian, $\hat{X}$ can be any operator and its time development $\hat{X}(t)$, it is well known that the time dependent operators for position $\hat{x}(t)$ and $\hat{y}(t)$ are obtained analytically as,

$$\hat{x}(t) = \hat{x} + \frac{\hat{p}}{qB} \cos \omega t + \frac{\hat{p}}{qB} \sin \omega t,$$

$$\hat{y}(t) = -\frac{\hat{p}}{qB} + \frac{\hat{p}}{qB} \cos \omega t - \frac{\hat{p}}{qB} \sin \omega t,$$

where $\omega = qB / m$ is the cyclotron frequency, $\hat{p}_x = -i\hbar \partial_\hat{y}$ is the $y$-component of the momentum operator and $\hat{p}_y = -i\hbar \partial_\hat{y}$ is the $x$-component of the momentum operator. The expectation value of the position $\langle \hat{x}(t) \rangle$ and $\langle \hat{y}(t) \rangle$ are obtained by implementing the initial condition for wavefunction at $r = r_0$ with $r_0$ being the initial center of wavefunction $\psi$ is given by

$$\psi(r,0) = \frac{1}{\sqrt{\pi \sigma_y}} \exp \left\{ -\frac{(r-r_0)^2}{2\sigma_y^2} + ik_0 \cdot r \right\},$$

where the magnetic length $\sigma_y$ is the initial standard deviation, and $ik_0$ is the initial canonical momentum.

A particle in a uniform magnetic field can be solved straightforwardly, however for the case of non-uniform electric and magnetic field cases, the theoretical derivations become long and complicated.

For the numerical calculations, a two-dimensional Schrödinger equation code is developed and the calculations are done on a GPU (Nvidia GTX-980: 2048 cores/4GB @1.126GHz), using CUDA. Successive over relaxation (SOR) scheme for time integration is in use in the numerical calculation. Furthermore, the numerical errors had removed from the numerical calculation by subtracting the variances in uniform magnetic field to non-uniform magnetic field. By using the finite difference method in space with Crank-Nicolson scheme for the time integration, the Schrödinger equation above become as

$$\left( I - \frac{\Delta t}{2i\hbar} \hat{H} \right) \{\psi^{n+1}\} = \left( I + \frac{\Delta t}{2i\hbar} \hat{H} \right) \{\psi^n\},$$

where $I$ is a unit matrix, $\hat{H}$ the numerical Hamiltonian matrix, and $\{\psi^n\}$ stands for the discretized set of the two dimensional time-dependent wavefunction $\psi(x, y, t)$ at a discrete time $t_n = n\Delta t$ to be solved numerically.

The exact solution $\psi(r, t)$ for the two-dimensional Schrödinger with a uniform magnetic field with a Landau gauge, of $A_y = -By$, $A_x = 0$, $A_z = 0$, is shown as

$$\psi(r, t) = \frac{1}{\sqrt{4\pi\hbar^3 / \omega}} \exp \left\{ -\frac{1}{2\hbar^2} \left( \frac{y - u(t)}{\omega} \right)^2 \times \exp \left[ \frac{y^2 \sin 2\omega t}{2\hbar^2} - \frac{y^2 \sin 2\omega t}{2\hbar^2} \right] \right\},$$

where $\omega = qB / m$ is the magnetic length, the $\omega = qB / m$ is the cyclotron frequency, $y_0 = k_0 \ell_B$, and $u(t)$ is classical velocity of the particle in $x$-direction. By referring to equation above, it is obvious that the standard deviation, variance, or uncertainty, in position remain constant throughout the time. Therefore, in case of uniform magnetic field, $L_B = \infty$, $\sigma_y^2(t) = \ell_B^2 = \text{const}$.

2. Quantum mechanical uncertainty energy

2.1 Initial condition: Electrostatic potential due to a field particle

Here the field particle is a quantum-mechanical particle, whose center is assumed to be at the origin with the wavefunction $\psi_i$ is fixed in space and time, as

$$\psi_i(r) = \frac{\exp \left( -\frac{x^2 + y^2}{2\sigma_y^2} \right)}{\sqrt{\pi \sigma_y}} \times \frac{\exp \left( -\frac{z^2}{2\sigma_z^2} \right)}{\sqrt{\pi \sigma_z}},$$

where $\sigma_z$ is the variance in position in z-direction, i.e., along the magnetic field. In magnetically confined fusion plasmas, $\sigma_z - \hbar / m v_i \ll \sigma_y$ holds, so that the square of the second factor can be approximately the same as a Dirac delta function $\delta(z)$ centered at $z = 0$ . Thus, the electrostatic potential $V_e$ in the $x - y$ plane, due to the distributed charge is given by

$$V_e(R) = -\frac{\sigma_y}{4\pi \varepsilon_0 \sigma_y} \int_0^\infty R' e^{(R/R_0)^2} K(M) dR',$$

where $R = \sqrt{x^2 + y^2}$, $\sigma_y$ is an electric charge of the field particle, $\varepsilon_0$ is the vacuum permittivity, and $K(M)$ is the
2.2 Expectation values and variances

The time evolution of total energy \( E = K + U \), kinetic energy \( K = (mv^2)/2m \) and the potential energy \( U = qV \) is shown in Fig. 2.4 for initial speed of \( v_0 = 10 \), \( v_0 = 5 \) and \( v_0 = 1 \). In quantum mechanical point of view, kinetic energy is the sum of directional energy and uncertainty energy. Referring to Fig. 2.1(a), for high speed case, \( v_0 = 10 \) the particle shows the similar behavior to the classical one. Only small amount of directional energy is converted to uncertainty energy. However, for the slow particle cases, \( v_0 = 5 \) and \( v_0 = 1 \) it is shown in Fig. 2.1(b, c) that the directional energy is converted to uncertainty energy after some time. Comparing with the same amount of gyrations, this phenomenon is significant for slower particle cases.

In the case of high speed particle, \( v_0 = 10 \), the trajectory in the phase \( \{ r, p \} \) is similar to the classical one and the variance in momentum and position oscillate with almost constant amplitudes. However, the trajectories both in the momentum space and position in space are gradually decreasing in respective in cyclotron radii. In the meantime, the variance oscillations show significant increase in variance \( \Delta \sigma^2 \).

In the longer period of evolutions, this phenomenon has the similar outcome to the lower speed case, \( v_0 = 5 \). It is shown that significant decrement in radii with time for both trajectories in momentum space and the configuration space. At the same time, the variances in momentum and configuration space grow with time and reach saturation. The saturation stage reached when all the directional energy is being converted to uncertainty energy.
\[ A = -B^2 \left(1 - \frac{\dot{y}}{2L_n} + \frac{\dot{y}^2}{2} \right) \epsilon_s. \]

The grad-B drift velocity analytical solution is compared with the numerical calculation. Substituting the vector potential into the two-dimensional Schrödinger equation, the Hamiltonian \( \hat{H} \) for a charge particle with a mass \( m \) and a charge \( q \) in the absence of an electrostatic potential for the non-relativistic charge particle, is given to first order in \( L_n^1 \) as,

\[
\hat{H} = \frac{1}{2m} \left( \hat{p}_x^2 + \hat{p}_y^2 - \frac{\Delta^1_1}{qBL_n} \right),
\]

where

\[
\Delta^1_1 = \left( 1 + \frac{\hat{p}_x^2}{qBL_n} \right) qB \hat{y} + \left( 1 + \frac{\hat{p}_y^2}{2qBL_n} \right) \hat{p}_y.
\]

The exact mechanical momentum operator \( m \hat{v} = \hat{p} - qA = (\hat{m}, \hat{m}) \) is given as,

\[
\hat{m} \hat{v} = \hat{p}_x + qB \hat{y} \left( 1 - \frac{\dot{y}}{2L_n} \right),
\]

where \( \hat{v} \) is the velocity operator. The mechanical momentum operator \( \hat{m} \hat{v} \) to first order in \( L_n^1 \) is given as,

\[
\hat{m} \hat{v} = \hat{p}_x - \frac{3 \hat{p}_x^2}{2qBL_n}.
\]

From the Heisenberg equation of motion, the time derivative of \( \hat{P}_x \) is given as,

\[
\frac{d}{dt} \hat{P}_x = \left( 1 + \frac{\hat{p}_x}{qBL_n} \right) \hat{m} \hat{v} + \frac{\omega}{2} \hat{P}_x = \hat{\Omega} \hat{P}_x,
\]

which leads to the definition of the angular frequency operator as,

\[
\hat{\Omega} = \left( 1 + \frac{\hat{p}_x}{qBL_n} \right) \omega.
\]

On the other hand, for the momentum operator in the y-direction \( \hat{P}_y \) we have,

\[
\frac{d}{dt} \hat{P}_y = -\omega \left( \hat{P}_x - \frac{3 \hat{P}_x^2}{2qB^2L_n} \right).
\]

To derive the time dependent momentum operators \( \hat{P}_x(t) \) and \( \hat{P}_y(t) \) with Heisenberg equation of motion, the equation above are expanded using the Heisenberg picture,

\[
\hat{X}(t) = \exp \left( -\frac{t}{\hbar} \hat{H} \right) \hat{X} \exp \left( \frac{t}{\hbar} \hat{H} \right).
\]

Let us choose the operator \( \hat{X} \) as

\[
\hat{X} = \hat{P}_x,
\]

and

\[
\hat{X} = \hat{P}_x + \frac{6 \hat{P}_x^2 - 9 \hat{P}_x^3}{2qBL_n}.
\]

Using Heisenberg equation of motion, the time derivative of the operators given in can be obtain as,

\[
\hat{P}_x(t) = \frac{3 \hat{P}_x^2 + 3 \hat{P}_x^3}{4qBL_n} + \hat{P}_x \left( \hat{P}_x - \frac{2 \hat{P}_x^2 + 3 \hat{P}_x^3}{4q^2B^2L_n} \right) \cos \hat{\Omega} t
\]

\[
+ \left( \frac{\hat{p}_x + \hat{P}_x \hat{P}_y + \hat{P}_x \hat{P}_y}{qB} \right) \sin \hat{\Omega} t - \frac{\hat{p}_x - \hat{P}_x \hat{P}_y - \hat{P}_x \hat{P}_y}{qB} \cos 2 \hat{\Omega} t.
\]

Using the results above, the time dependent operator of position \( \hat{x}(t) \) and \( \hat{y}(t) \) are obtained. From operator \( \hat{P}_x \), the time dependent operator \( \hat{x}(t) \) is found and shown, to first order in \( L_n^1 \) as,

\[
qB \hat{y} = \left( 1 - \frac{\hat{P}_x}{qBL_n} \right) \hat{P}_x \left[ \frac{3 \hat{P}_x^2 + 3 \hat{P}_x^3}{4q^2B^2L_n} + \frac{\hat{P}_x \left( 2 \hat{P}_x^2 + 3 \hat{P}_x^3 \right)}{4q^2B^2L_n} - \frac{\hat{p}_x \hat{P}_x}{q^2B^2L_n} \right] \cos \hat{\Omega} t
\]

\[
+ \left( \frac{\hat{p}_x + \hat{P}_x \hat{P}_y + \hat{P}_x \hat{P}_y}{qB} \right) \sin \hat{\Omega} t + \frac{\hat{p}_x - \hat{P}_x \hat{P}_y - \hat{P}_x \hat{P}_y}{q^2B^2L_n} \cos 2 \hat{\Omega} t.
\]

On the other hand, the time dependent operator \( \dot{x}(t) \) is obtained by integrating \( \dot{u}(t) \) form with time \( t \),

\[
\dot{x}(t) = \dot{x} + \int_0^t \dot{u}(t) \, dt,
\]

which leads to,
\[ \dot{x}(t) = \dot{x} + \left( \frac{1}{qB} - \frac{\dot{P}_y}{q B^2 L_n'} \right) \dot{P}_y + \frac{\dot{\Pi}_L + \dot{\Pi}_L + \dot{\Pi}_L}{4q^2 B^2 L_n'} + \left( \frac{\dot{P}_y}{q B} - \frac{\dot{\Pi}_y}{q^2 B^2 L_n'} \right) \cos \Omega t \]

\[ + \left( \frac{2\dot{P}_x + \dot{\Pi}_y}{q B} \right) \sin \Omega t + \left( \frac{\dot{P}_y + \dot{\Pi}_y}{4q^2 B^2 L_n'} \right) \cos 2\Omega t + \left( \frac{\dot{P}_y - \dot{\Pi}_y}{4q^2 B^2 L_n'} \right) \sin 2\Omega t . \]

From the operator \( m \dot{u} \), together with the operator \( \dot{\Pi}_L(t) \), the time dependent momentum operator \( m \dot{u}(t) \) along x-axis is obtained as

\[ m \dot{u}(t) = \frac{\dot{P}_x + \dot{\Pi}_y}{2q B L_n'} + \left( \frac{\dot{P}_x - \dot{\Pi}_y}{2q B L_n'} \right) \cos \Omega t + \left( \frac{\dot{P}_y + \dot{\Pi}_y}{2q B L_n'} \right) \sin \Omega t + \left( \frac{\dot{P}_y - \dot{\Pi}_y}{2q B L_n'} \right) \sin 2\Omega t . \]

### 3.2 Time average of variance

By solving Heisenberg equation of motion, the time averages over cyclotron period of variance in position \( \sigma^2(t) \) and total momentum \( \sigma^2(t) \) to zeroth order in \( L_n' \), are given as

\[ \sigma^2(t) = \sigma^2(t) + \sigma^2(t) = \frac{7}{4} \hbar + \frac{5}{4} \hbar B , \]

\[ \sigma^2_m(t) = \sigma^2_m(t) + \sigma^2_m(t) = \frac{3}{4} \hbar B + \frac{3}{4} \hbar B , \]

\[ \sigma^2_p(t) = \sigma^2_p(t) + \sigma^2_p(t) = \frac{1}{2} \hbar B + \frac{3}{2} \hbar B , \]

using \( \sigma^2_p = \sigma_p^2 = \hbar / 2B \), \( \sigma^2_p = \sigma_p^2 = \sigma^2_p = \hbar B / 2 \), and \( \sigma^2_m = \sigma_m^2 = \sigma^2_m = \hbar B \) for the initial wavefunction when gradient scale length \( L_n = \infty \). It is noted that \( \sigma_p(t) = \sigma_m(t) \) since \( \dot{P}_x = m \dot{u} \) is due to the Landau gauge \( A = A(y) e_x \) adopted in this study.

### 3.3 Expansion rates of variance

By solving Heisenberg equation of motion, the analytical solution of expansion rates of variance in position, mechanical momentum, and total momentum to the first order in \( L_n' \), are given as

\[ \frac{d \sigma^2(t)}{dr} = \frac{d \sigma^2(t)}{dr} + \frac{d \sigma^2(t)}{dr} = \frac{3}{2} \hbar B + \frac{1}{2} \hbar B , \]

\[ \frac{d \sigma^2_m(t)}{dr} = \frac{d \sigma^2_m(t)}{dr} + \frac{d \sigma^2_m(t)}{dr} = \frac{1}{2} \hbar B + \frac{1}{2} \hbar B , \]

\[ \frac{d \sigma^2_p(t)}{dr} = \frac{d \sigma^2_p(t)}{dr} + \frac{d \sigma^2_p(t)}{dr} = 0 + \frac{1}{2} \hbar B . \]

### 3.4 Grad-B drift velocity

Since the time dependent momentum operator \( m \dot{u}(t) \) is obtained, the grad-B drift operator is obtained straightforwardly as \([3]\),

\[ \dot{u}_v_B = \dot{u}(t) = \frac{\dot{P}_x + \dot{\Pi}_y}{2m q B L_n'} + \frac{H}{q B L_n'} + O(L_n'^2) . \]

The expectation value of the grad-B drift velocity operator \( u_{vB} = \langle \dot{u}_v_B \rangle \) is given as follows

\[ u_{vB} = \frac{1}{\pi \sigma_B^2} \int e^{-\frac{(r-n)^2}{2\sigma^2}} u_{vB} e^{-\frac{(r-n)^2}{2\sigma^2}} d^2r = \frac{m v_B^2}{2q B L_n'} + \frac{1}{2q B L_n'} \left( \frac{\sigma^2}{m \sigma_p^2 + m \sigma^2} \right) , \]

where \( m v_B = \langle m \dot{u} \rangle = \langle \dot{P}_x - q \dot{A} \rangle \). When we use the magnetic length of \( \sqrt{\hbar / q B} \) as \( \sigma_y \), then we have

\[ u_{vB} = \frac{m v_B^2}{2q B L_n'} + \frac{3 \hbar}{4m B} . \]

Note that the first term of \( u_{vB} \) coincides with the classical formula for the grad-B drift and the second term represents the quantum mechanical drift due to the uncertainty.

### 3.5 Summary

The grad-B drift velocity of a charged particle is solved with considering quantum mechanical effect by using the Heisenberg equation of motion. It is shown that the grad-B drift velocity operator obtained in this study agrees with the classical counterpart, when the uncertainty is ignored. The time evolution of the position and momentum operators are also analytically obtained for the non-relativistic spinless charged particle.

The theoretical derivation solutions do agree with the numerical results on the grad-B drift velocity in the presence of a non-uniform magnetic field. The quantum mechanical grad-B drift velocity formulations clearly show the drift velocity dependence on mass \( m \) and gradient scale length \( L_n \). The result implies that light particles with low energy would drift faster than classical drift theory predicts.

### 4. Quantum mechanical \( \mathbf{E} \times \mathbf{B} \) drift

#### 4.1 Initial condition

In the presence of a weakly non-uniform magnetic field \( B = B(1 - y / L_n) e_y \), a Landau gauge-like quadratic vector potential is given as

\[ A = -By \left( 1 - \frac{y}{2L_n} \right) e_z , \]

and with the presence of a weakly non-uniform electric field \( E = E(1 - y / L_n) e_y \), a scalar potential is given as

\[ V = -Ey \left( 1 - \frac{y}{2L_n} \right) . \]

#### 4.2 Time dependent operators for the non-uniform electric and magnetic field

Substituting the vector potential and the electrostatic potential into the two-dimensional Schrödinger equation, the Hamiltonian \( H \) for a charge particle with a mass \( m \) and a charge \( q \) for a non-relativistic charge particle, to first order in
\( L_w^3 \) and \( L_w^{-3} \) is given as
\[
\hat{H} = \frac{1}{2m} \left( \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 - \frac{1}{qBL_n} \right) + \Delta \hat{H},
\]
where \( \hat{p}_x = -i\hbar \nabla_x \) is the \( x \)-component of the momentum operator and \( \hat{p}_z = -i\hbar \nabla_z \) is the \( z \)-component of the momentum operator defined in operator \( \hat{\Pi}_x \) as,
\[
\hat{\Pi}_x = \left( 1 + \frac{\hat{p}_x}{2qBL_n} - \frac{3mE}{2qBL_n} B + \frac{1}{qBL_n} \right) qB y\hat{B}
+ \left( 1 + \frac{\hat{p}_x}{2qBL_n} - \frac{1}{2qBL_n} \right) \left( \hat{p}_x - \frac{mE}{B} \right),
\]
and \( \Delta \hat{H} \) is given as,
\[
\Delta \hat{H} = \left[ \frac{\hat{p}_x - \frac{1}{2} mE}{B} \right] \left[ \frac{1}{2} mE \right] - \left( \hat{p}_x - \frac{mE}{B} \right) E.
\]
It is assumed that \( L_w^3 \) is of the same order as \( L_w^{-3} \).

4.3 Quantum mechanical \( E \times B \) drift velocity

By solving the Heisenberg Equation of motion, the expectation value of drift velocity is obtained as [2],
\[
\langle \dot{u}_{\text{drift}} \rangle = \frac{mv_y^2}{2qBL_n} + \frac{1}{2qBL_n} \frac{\hbar^2}{\sigma_B^2} + \frac{m \sigma_B^2}{2},
\]
where \( \sigma_B = \left\{ \sigma_B \right\} \) \( \left\{ \hat{p} - q \hat{A} \right\} \). When we use the magnetic length of \( \sqrt{\hbar / qB} \) as \( \sigma_B \), then we have
\[
\dot{u}_{E-B} = \frac{mv_y^2}{2qBL_n} + \frac{3\hbar}{4mL_n} y_d E_B,
\]
\[
\dot{u}_{E-B} = \left[ 1 + \frac{\mu_0}{qBL_n} - \frac{mE}{2qBL_n} B \right] \left( \hat{p}_x - \frac{mE}{B} \right) E.
\]

The drift velocity as shown is due solely to the magnetic field non-uniformity. The drift velocity due to the electric and magnetic field non-uniformity, \( \dot{u}_{E-B} \) is the same as the classical drift velocity to first order in \( L_w^3 \) and \( L_w^{-3} \), which can be easily derived from the classical equation of motion.

4.4 Summary

Using the time dependent operators, it is shown how the variances in position and momentum grow with time. The theoretical solutions are generally consistent with the numerical results on the expansion rates in position and momenta in the presence of a weakly non-uniform electric and magnetic field. It is important to note that the expansion rates do not depend on the strength of the electric field \( E \) and its gradient scale length \( L_g \) to first order in \( L_w^3 \). It is also shown that the drift velocity operator agrees with the classical counterpart.

5. Quantum mechanical \( E \times B \) drift at higher order of electromagnetic field inhomogeneity

5.1 Initial condition

The two-dimensional Schrödinger equation is solved in the presence of an inhomogeneous magnetic field \( B = B_0 \left( 1 - y / L_E \right) e_z \), in which a Landau gauge-like quadratic vector potential is given as
\[
A = -B_0 y \left( 1 - \frac{y}{2L_E} \right) e_z,
\]
and with the presence of an inhomogeneous electric field \( E = E_0 \left( 1 - y / L_E \right) + y^2 / L_E^2 e_y \), a quadratic scalar potential is given as
\[
V = -E_0 y \left( 1 - \frac{y^2}{2L_E^2} + \frac{3y^2}{3L_E^2} \right).
\]

5.2 Time dependent operators for the inhomogeneous electric and magnetic field

The Hamiltonian \( \hat{H} \) for a charge particle with a mass \( m \) and a charge \( q \) without the electrostatic potential for the non-relativistic charge particle, to first order in \( L_w^3 \), \( L_w^{-3} \), and \( \epsilon_E^2 \) as
\[
2\hat{m} \hat{H} = \hat{P}_z \hat{\Pi}_y - \hat{\Pi}_y \hat{P}_z = \left( \frac{1}{qBL_n} + \frac{2}{3q^2B^2 \epsilon_E} \right) \frac{mE}{B} + \Delta \hat{H},
\]
and
\[
\Delta \hat{H} = \left( \hat{p}_x - \frac{mE}{2B} \right) \left( \hat{p}_x - \frac{mE}{B} \right) E.
\]
where \( \hat{p}_x = -i\hbar \nabla_x \) is the \( y \)-component of the momentum operator and \( \hat{p}_z = -i\hbar \nabla_z \) is the \( z \)-component of the momentum operator defined in operator \( \hat{\Pi}_y \) as,
\[
\hat{\Pi}_y = \left[ 1 + \frac{3}{2qBL_n} + \frac{1}{q^2B^2 \epsilon_E^2} \right] \left( \hat{p}_x - \frac{mE}{B} \right) E.
\]

5.3 Quantum mechanical \( E \times B \) drift velocity

By solving Heisenberg equation of motion, the drift velocity is obtained as,
\[
\dot{u}_{\text{drift}} = \frac{mv_y^2}{2qBL_n} + \frac{3\hbar}{4mL_n} y_d E_B,
\]
where the grad-\( B \) drift velocity and the \( E \times B \) drift velocity are shown as [1].
Due to the inhomogeneity of magnetic field, \( u_{cb} \) is found. The first term of the drift velocity is the classical grad-B drift velocity, which this coincides with the classical drift velocity, and the second term \( 3\hbar/4ml_B^2 \) represents the quantum mechanical grad-B drift velocity due to the uncertainty. This drift velocity is the same as shown in Chap. 3 however, in the absence of the inhomogeneous electric field.

In the presence of the both inhomogeneous electric and magnetic field, \( u_{E*B} \) is found. The first term of the drift velocity is the classical \( E^*B \) drift velocity due to the homogeneous electric field, which this coincides with the classical drift velocity and can be easily derived from the equation of motion. The second term \( 5\hbar E/4B^2\ell^2 E \) represents the quantum mechanical \( E^*B \) drift velocity due to the uncertainty.

The quantum mechanical \( E^*B \) drift velocity formulations clearly show the drift velocity dependence on electric field \( E \), magnetic field \( B \), charge \( q \), and gradient scale length \( \ell_E \). However, the drift velocity is independent of gradient scale length \( L_B \).

5.4 Summary

In the presence of a weakly inhomogeneous electric and magnetic field, the \( E^*B \) drift velocity operator is obtained. It is also shown that the drift velocity operator agrees with the classical counterpart. The quantum mechanical part of the drift velocity formula, \( u_{QM} = 3\hbar/4ml_B^2 + 5\hbar E/4B^2\ell^2 E \), clearly shows the dependence on mass \( m \), charge \( q \), magnetic field \( B \), electric field \( E \), and gradient scale lengths both \( L_B \) and \( \ell_E \). However, the drift velocity \( u_{QM} \) is independent of the gradient scale length \( L_B \). The result implies that, with the effect of an inhomogeneous electric and magnetic field, low energy light particles would drift faster than classical drift theory predicts.

6. Conclusion

It is analytically shown that the variance in position reaches the square of the interparticle separation, which is the characteristic time much shorter than the proton collision time of plasma fusion. After this time the wavefunctions of the neighboring particles would overlap, thus the conventional classical analysis may lose its validity. The expansion time in position implies that the probability density function of such energetic charged particles expands fast in the plane perpendicular to the magnetic field and their Coulomb interaction with other particles becomes weaker than that expected in classical mechanics.

Previously, quantum mechanical approach in plasma studies are mainly concentrated on the low density and temperature cases. Therefore, here we would like to highlight that quantum-mechanical analyses are necessary even for fast charged particles if their long-time behavior is concerned in the presence of the electromagnetic field.

References (list of publications)